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SHOCK-WAVE STRUCTURE BASED ON IKENBERRY-TRUESDELL
APPROACH TO KINETIC THEORY OF GASES

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SUMMARY

A brief review of the Ikenberry-Truesdell method of solution of the Boltzmann equation is given, in which emphasis is placed upon the procedure called Maxwellian iteration. Corrections to the various iterations are made. This approach is then applied to the problem of shock-wave structure. A series solution of the type used by Grad for his equations and by Talbot and Sherman for the Chapman-Enskog equations is used to find the velocity and temperature profiles for a steady, plane, shock wave in an ideal gas of Maxwellian molecules. The results are not significantly different from the Navier-Stokes solution of the same case. The advantage of the present method of solution of the transfer equation lies in the fact that the form of the distribution function need not be specified.

INTRODUCTION

The Boltzmann integrodifferential equation forms the basis of the classical kinetic theory of gases. Its modern, rigorous derivation is the work of Kirkwood and Grad has discussed its validity. Both of these approaches are well discussed in references 1 and 2 and a more elementary treatment is given by Patterson in reference 3. The unknown function in the Boltzmann equation is the distribution function $F(\vec{x},\vec{\xi},t)$ which gives the number of molecules per unit volume of phase space at position $\vec{x}$ and time t with velocity $\vec{\xi}$. Except for the simple Maxwell distribution function, only approximate solutions of the Boltzmann equation have ever been found.

Actually, for most problems in aerodynamics a knowledge of the distribution function yields more information than is necessary, since only the gross properties of the gas are wanted and can be given any experimental significance. These gross or macroscopic properties are the absolute temperature T, density ρ, and stresses p_{ij}, p_{ijk}, . . . , $p_{ijk\ldots m}$ which are moments of either the absolute molecular velocity or the relative molecular velocity taken with respect to the distribution function over all of velocity space. The use of these moments and the derivation of the differential equations which they satisfy is called the Lagrangian formulation by Mott-Smith (ref. 4).

Instead of attempting to solve the Boltzmann equation itself in this Lagrangian approach, the Boltzmann equation is multiplied by a function of the molecular velocity components and integrated over velocity space. The result is Enskog's equation of transfer which is a generalization of a similar equation developed by Maxwell; it is also called the general "equation of change" (ref. 2). In the original derivation by Maxwell of the moment equations as well as in the subsequent method of Chapman and Enskog an explicit form of the distribution function F was assumed and this function was a solution in a certain sense of the Boltzmann equation itself. Grad (ref. 5) also used the same distribution function although he interpreted it differently; he indicated too how the equations might be derived without knowing F explicitly.

This idea of obtaining the equations relating the Lagrangian moments from the transfer equation without specifying the distribution function has been carried through in its ultimate form by Ikenberry and Truesdell (ref. 6). Although their paper is well written, it is long and contains a great deal of mathematical proof and development which, although essential to the derivation, is not necessary in order to develop the results. As the authors point out, the ideas are simple and the calculations, although long and elaborate, do not involve any advanced mathematics other than a knowledge of formal operation with tensors. Hence the present paper will give the formal development of the basic equations, and then use the method which Ikenberry and Truesdell call Maxwellian iteration to obtain the equations of one-dimensional steady flow which will be solved by Grad's method of series for the plane, steady shock-wave structure. Reference 6 contains eight different methods of iteration but only one is used here; it is more completely discussed than the others and is the first method given in reference 6 and thus it is necessary to use only the results of the first chapter of that reference. Also there are a few mistakes in the expressions for the collision integrals and some of the iterations in reference 6 will be presented in correct form herein.

The molecular model used is that of Maxwell, in which the molecules are point centers of force repelling each other as the inverse fifth power of the distance between molecules. The shock-wave structure problem is the only one solved since it does not involve boundary conditions, which would require a knowledge of the distribution function if they are to be formulated. The distribution function is not known so no explicit form for it is given anywhere in this paper.

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SYMBOLS

A,B,C,D	constant coefficients in shock-wave solution
B_1,B_2,B_4	constants taken from equation (7.22) of reference 6
b	distance from first molecule to initial asymptote of second molecular path in a collision
$C(Q)$	collision integral
c	magnitude of intrinsic velocity vector $\vec{c}$
$\vec{c}$	intrinsic velocity vector, $\vec{c} = c_1,c_2,c_3 = \vec{\xi} - \vec{u}$
c_p,c_v	specific heats at constant pressure and constant volume
E	divergence of velocity, $\dfrac{\partial u_i}{\partial x_i} = u_{i,i}$
E_{ij}	deviatoric rate of deformation, $\frac{1}{2}\left(u_{i,j} - u_{j,i}\right) - \frac{1}{3} E\delta_{ij}$
e	specific internal energy, c_vT
$F(\vec{x},\vec{\xi},t)$	molecular distribution function
G	constant in inverse fifth-power force law
K	constant ratio of viscosity to temperature
k	Boltzmann's constant
$L(Q)$	differential transfer operator; also, length parameter in shock-wave solution
M	Mach number of flow
m	mass of molecule; also, constant of integration in shock-wave solution

$n(\vec{x},t)$	number density of molecules
P	constant of integration in shock-wave problem
P_{ij}	pressure deviator or excess pressure tensor
Pr	Prandtl number of gas, $\mu c_p/\lambda$
$P_{2r\|s}$	spherical moment of order $q = 2r + s$, $\rho\overline{Y}_{2r\|s}$
$P_{2r\|} = P_{2r\|0}, P_s = P_{0\|s}$ where $s = i_1 i_2 \ldots i_s$	
p	static pressure
p_{ij}	second moment of $F(\vec{\xi})$, pressure tensor
$p_{ij\ldots n}$	higher moments of $F(\vec{\xi})$ with respect to intrinsic velocity $\vec{c}$
Q	constant of integration in shock-wave problem
q	order of moment, $2r + s$
$\vec{q} = q_1, q_2, q_3$	heat-flux vector
R	specific gas constant, k/m
r	intermolecular distance; also, power of c^2 in $Y_{2r\|s}$
s	degree of homogeneous spherical harmonic
T	absolute temperature
t	time; also, dimensionless temperature in shock problem
u	x component of $\vec{u}$
$\vec{u} = u_1, u_2, u_3$	velocity of gas, the average molecular velocity, $\vec{\xi}$
$2\vec{v}$	relative molecular velocity before encounter
w	dimensionless velocity in shock-wave problem
$2\vec{w}$	specific momentum of molecular pair

x	coordinate x_1
$\vec{x} = x_1, x_2, x_3$	position vector in three dimensions
Y	particular value of dimensionless coordinate η
Y_s	homogeneous spherical harmonic of order $s = i_1 i_2 \ldots i_s$
$Y_{2r\|s}(\vec{c}) = c^{2r} Y_s(\vec{c})$	
y	coordinate x_2; also, dimensionless form of x coordinate
z	coordinate x_3
α	ratio of thermal conductivity to viscosity, $\frac{15}{4} R$
β	dimensionless form of specific entropy
γ	ratio of specific heats, c_p/c_v
δ_{ij}	Kronecker delta, $\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$
$\delta_m{}^w$	maximum slope thickness of shock wave
ϵ	angle between plane of $\vec{v}$ and $\vec{v}'$ and a reference plane through $\vec{v}$; also, a shock-strength parameter
η	specific entropy of monatomic gas; also, dimensionless coordinate in shock-wave problem
θ	encounter angle related to ϕ
Λ_i	mean free path of gas upstream of shock wave
λ	coefficient of thermal conduction of gas, $\alpha\mu$
μ	coefficient of viscosity of the gas, KT
$\vec{\xi} = \xi_1, \xi_2, \xi_3$	absolute velocity vector of molecule
ρ	density of gas, mn

τ dimensionless temperature

ϕ angle between $\vec{v}'$ and $\vec{v}$ in a molecular collision, $\pi - 2\theta$

$$\int_{\infty} \cdot \cdot \cdot d\vec{\xi} = \int_{-\infty}^{\infty} d\xi_3 \int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{\infty} \cdot \cdot \cdot d\xi_1$$

$|$ separates indices $2r$ and s on functions $Y_{2r|s}$ and $P_{2r|s}$

$(\bar{\ })$ mean or expected value defined by equation (2)

$(\vec{\ })$ vector

Subscripts:

0,1,2, . . . order of approximation in series

i,f initial and final states for shock-wave flow

,i partial derivative with respect to x_i

2r indicates factor c^{2r} in polynomial

s s indices $i_1 i_2 \ldots i_s$

* reference condition in shock wave corresponding to $M = 1$

( ) around subscripts indicate sum over $s!$ permutations of s indices divided by $s!$

Superscripts:

$(fgh)^{\cdot}$ or $\dot{f}$ hydrodynamic or material derivative, $\frac{D}{Dt}$

$(r) = (0),(1),(2), \ldots$ order of iteration

* second of a pair of molecules

' values after molecular collision; also, ordinary derivative with respect to x or η

EXACT RELATIONS BETWEEN MOMENTS AND EQUATION OF TRANSFER

For a moderately rarefied monatomic gas the number density of molecules at position $\vec{x}$ and time t with velocity $\vec{\xi}$ is given by the distribution function $F(\vec{x},\vec{\xi},t)$. The number of molecules per unit volume of $\vec{x}$ space is then the number density

$$n(\vec{x},t) \equiv \int_{\infty} F(\vec{x},\vec{\xi},t)\,d\vec{\xi} \tag{1}$$

where the symbol $\int_{\infty} \ldots d\vec{\xi}$ represents the triple integral $\int_{-\infty}^{\infty} d\xi_3 \int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{\infty} \ldots d\xi_1$. The mass density or, simply, density of the gas is then $\rho = mn$ where m is the mass of a single molecule. Only simple gases will be considered; therefore, m is a constant in this paper. The mean or expected value of any function $Q(\vec{x},\vec{\xi},t)$ is defined by

$$n\overline{Q} \equiv \int_{\infty} QF\,d\vec{\xi} \tag{2}$$

In particular, if $Q = \xi_i$,

$$u_i \equiv \overline{\xi}_i = \frac{1}{n}\int_{\infty} \xi_i F\,d\vec{\xi} \tag{3}$$

are the components of the average or mass velocity $\vec{u}$.

In the Lagrangian formulation all additional mean values are just those of the various products of the components of the intrinsic or peculiar velocity defined as $\vec{c} \equiv \vec{\xi} - \vec{u}$ or

$$c_i \equiv \xi_i - u_i \tag{4}$$

Then the nth moment of F is defined as

$$p_{i_1 i_2 \ldots i_n} \equiv \rho\overline{c_{i_1} c_{i_2} \ldots c_{i_n}} = m\int_{\infty} c_{i_1} c_{i_2} \ldots c_{i_n} F\,d\vec{\xi} \tag{5}$$

Because of the definition of c_i the first moment $p_i = \rho\overline{c_i} = m\int_\infty c_i F\, d\vec{\xi} = 0$; the second moment

$$p_{ij} = \rho\overline{c_i c_j} = m\int_\infty c_i c_j F\, d\vec{\xi} \tag{6}$$

is the pressure tensor; one-half of the contracted third moment

$$q_i \equiv \frac{1}{2}\, p_{ijj} = \frac{m}{2}\int_\infty c_i c_j c_j F\, d\vec{\xi} \tag{7}$$

is the flux of energy or heat-flux vector; and the third- and higher-order moments have no special names.

The contraction of the pressure tensor gives three times the scalar pressure

$$3p \equiv p_{ii} = m\int_\infty c^2 F\, d\vec{\xi} \tag{8}$$

Thus if the absolute temperature is defined by

$$RT = \frac{p}{\rho} \tag{9}$$

where R is the gas constant per unit mass, the temperature can be written as

$$T = \frac{1}{3nR}\int_\infty c^2 F\, d\vec{\xi}$$

Also a divergenceless pressure tensor which Ikenberry and Truesdell call the pressure deviator is defined as

$$P_{ij} \equiv p_{ij} - p\delta_{ij} \tag{10}$$

so that the contraction $P_{ii} = 0$.

Although both the vector notation and Cartesian tensor notation have been used so far, the latter will, in general, be adhered to as much as possible. This tensor notation includes the use of the double summation convention $p_{ii} \equiv \sum_{i=1}^{3} p_{ii}$, $p_{ikkll} \equiv \sum_{k=1}^{3}\sum_{l=1}^{3} p_{ikkll}$, and so forth. No special form for F has been assumed or will be necessary. Only the existence and differentiability of the moments are going to be required.

The equation of transfer will not be derived since it can be taken from reference 1 (p. 209), reference 3 (p. 20), reference 5 (p. 362), or the well-known treatise of Chapman and Cowling. In the notation used by Ikenberry and Truesdell this equation is written in the symbolic form

$$L(Q) = mC(Q) \tag{11}$$

where

$$L(Q) \equiv (\rho\overline{Q})^{\boldsymbol{\cdot}} + \rho\overline{Q}E + u_{j,i}\rho\overline{c_i \frac{\partial Q}{\partial c_j}} - p_{ij,i}\overline{\frac{\partial Q}{\partial c_j}} + \left(\rho\overline{c_i Q}\right)_{,i} \tag{12}$$

and

$$C(Q) \equiv \int_\infty d\vec{\xi}\int_\infty d\vec{\xi}^{*}\int_0^\infty b\, db\int_0^{2\pi} d\epsilon\ \delta QFF^{*}v \tag{13}$$

Much new notation has been introduced here. In equation (12) the dot denotes the hydrodynamic derivative

$$(\ )^{\boldsymbol{\cdot}} \equiv \frac{D(\ )}{Dt} \equiv \frac{\partial(\ )}{\partial t} + (\ )_{,i}u_i \tag{14}$$

the subscript comma denotes the partial derivative with respect to x_i, and

$$E \equiv \frac{\partial u_i}{\partial x_i} = u_{i,i} \tag{15}$$

is the divergence of the velocity. The momentum equation

$$\rho\dot{u}_i + p_{ij,j} = 0 \tag{16}$$

which normally follows from equation (11) when $Q = c_i$ will not do so here since it has been used to eliminate $\dot{u}_i$. In equation (13) an asterisk denotes the second of a pair of molecules, the prime denotes the outcome of a collision, and b and ϵ are standard collision parameters whose precise nature need not be specified since the integrals involving them have been evaluated in reference 6 and only the results will be needed here. Also,

$$\delta Q = Q^{*\prime} + Q' - Q^* - Q = 2\left[(Q^*)' - Q^*\right] \tag{17}$$

and

$$\left.\begin{aligned} 2\vec{v} &= \vec{\xi}^* - \vec{\xi} = \vec{c}^* - \vec{c} \\ 2\vec{w} &= \vec{c}^* + \vec{c} \end{aligned}\right\} \tag{18}$$

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Conservation of momentum and energy during a collision requires that

$$\left.\begin{aligned} \vec{w} &= \vec{w}' \\ v &= v' \end{aligned}\right\} \tag{19}$$

if $v = |\vec{v}|$ and $v' = |\vec{v}'|$.

The apparent choice for Q would seem to be the various products in c which would lead to expressions for equations (12) and (13) in terms of the moments $p_{i_1 i_2 \ldots i_n}$. But Ikenberry and Truesdell point out that this choice would lead to unnecessary complications which can be avoided by taking for $Q(\vec{c})$ homogeneous spherical harmonics in $\vec{c}$ as was originally suggested by Maxwell. Their choice of spherical harmonics is the set of symmetric functions proposed by Ikenberry (ref. 7):

$$\left.\begin{aligned} Y(\vec{c}) &= 1 \\ Y_i(\vec{c}) &= c_i \\ Y_{ij}(\vec{c}) &= c_i c_j - \frac{1}{3} c^2 \delta_{ij} \\ Y_{ijk}(\vec{c}) &= c_i c_j c_k - \frac{3}{5} c^2 c_{(i}\delta_{jk)} \\ Y_{ijkl}(\vec{c}) &= c_i c_j c_k c_l - \frac{6}{7} c^2 Y_{(ij}(\vec{c})\delta_{kl)} - \frac{1}{5} c^4 \delta_{(ij}\delta_{kl)} \\ Y_{ijklm}(\vec{c}) &= c_i c_j c_k c_l c_m - \frac{10}{9} c^2 Y_{(ijk}(\vec{c})\delta_{lm)} - \frac{3}{7} c^4 Y_{(i}(\vec{c})\delta_{jk}\delta_{lm)} \end{aligned}\right\} \tag{20}$$

and so forth, where parentheses around s subscripts indicate a sum over the $s!$ permutations of the indices divided by $s!$. Thus, for example:

$$6Y_{(ij}\delta_{kl)} = Y_{ij}\delta_{kl} + Y_{ik}\delta_{jl} + Y_{il}\delta_{kj} + Y_{jk}\delta_{il} + Y_{jl}\delta_{ik} + Y_{kl}\delta_{ij}$$

and because both Y_{ij} and δ_{kl} are symmetric, the 24 terms reduce to 6. The $Y_s(\vec{c})$ thus defined, s representing a set of s indices $i_1, i_2 \ldots i_s$, are the components of symmetric tensors which when contracted on any pair of indices reduce to zero.

Let

$$Y_{2r|s}(\vec{c}) \equiv c^{2r} Y_s(\vec{c}) \tag{21}$$

and define the spherical moment $P_{2r|s}$ of order $2r + s = q$ by

$$P_{2r|s} \equiv \rho \overline{Y}_{2r|s} = m \int_{\infty} c^{2r} Y_s(\vec{c}) F \, d\vec{c} \tag{22}$$

where $d\vec{c} = d\vec{\xi}$ and the integration is over all of velocity $\vec{c}$ space. This moment of order q is a symmetric isotropic tensor function of the qth moments and when contracted on any pair of indices it becomes zero. It includes and generalizes the second-order pressure deviator defined in equation (10). The calculations which relate the spherical moments of equation (22) to the moments defined by equation (5) are set forth in appendix A and the results are the following expressions:

$$\left.\begin{aligned}
P &= \rho \\
P_i &= 0 \\
P_{ij} &= p_{ij} - p\delta_{ij} \\
P_{2|} &= 3p \\
P_{2|i} &= p_{ijj} = 2q_i \\
P_{ijk} &= p_{ijk} - \frac{3}{5} p_{ll(i}\delta_{jk)} \\
P_{ijkl} &= p_{ijkl} - \frac{6}{7} p_{mm(ij}\delta_{kl)} + \frac{3}{35} p_{mmnn}\delta_{(ij}\delta_{kl)} \\
P_{2|ij} &= p_{ijkk} - \frac{1}{3} p_{kkll}\delta_{ij} \\
P_{4|} &= p_{kkll} \\
P_{ijklm} &= p_{ijklm} - \frac{10}{9} p_{rr(ijk}\delta_{lm)} + \frac{5}{21} p_{rrss(i}\delta_{jk}\delta_{lm)} \\
P_{2|ijk} &= p_{llijk} - \frac{3}{5} p_{llmm(i}\delta_{jk)} \\
P_{4|i} &= p_{illmm} \\
P_{4|ij} &= p_{ijllmm} - \frac{1}{3} p_{kkllmm}\delta_{ij} \\
P_{6|} &= p_{kkllmm}
\end{aligned}\right\} \tag{23}$$

with the inverse relations

$$\left.\begin{aligned}
p_{ij} &= P_{ij} + p\delta_{ij} = P_{ij} + \tfrac{1}{3} P_{2|}\delta_{ij} \\
p_{ijk} &= P_{ijk} + \tfrac{3}{5} P_{2|(i}\delta_{jk)} \\
p_{ill} &= P_{2|i} = 2q_i \\
p_{ijkl} &= P_{ijkl} + \tfrac{6}{7} P_{2|(ij}\delta_{kl)} + \tfrac{1}{5} P_{4|}\delta_{(ij}\delta_{kl)} \\
p_{ijll} &= P_{2|ij} + \tfrac{1}{3} P_{4|}\delta_{ij} \\
p_{kkll} &= P_{4|} \\
p_{ijklm} &= P_{ijklm} + \tfrac{10}{9} P_{2|(ijk}\delta_{lm)} + \tfrac{3}{7} P_{4|(i}\delta_{jk}\delta_{lm)} \\
p_{ijkll} &= P_{2|ijk} + \tfrac{3}{5} P_{4|(i}\delta_{jk)} \\
p_{ikkll} &= P_{4|i} \\
p_{ijllmm} &= P_{4|ij} + \tfrac{1}{3} P_{6|}\delta_{ij} \\
p_{kkllmm} &= P_{6|}
\end{aligned}\right\} \qquad (24)$$

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These are the only moments which are used in this paper.

In the transfer equation the function Q is now taken equal to $Y_{2r|s}$ so that the next step is the calculation of the $L(Y_{2r|s})$ and $C(Y_{2r|s})$ terms. This is straightforward but increasingly lengthy as $q = 2r + s$ increases, insofar as $L(Y_{2r|s})$ is concerned, and the results are (as given in the original paper but with p_{ij} replaced by $P_{ij} + p\delta_{ij}$):

$$
\left.
\begin{aligned}
L(1) &= \dot{\rho} + \rho E \\
L(Y_i) &= 0 = P_{ij,j} + p_{,i} + \rho \dot{u}_i \\
L(c^2) &= 3\dot{p} + 5pE + 2P_{ij}E_{ij} + 2q_{i,i} \\
L(Y_{ij}) &= \dot{P}_{ij} + P_{ij}E + 2P_{k(i}u_{j),k} + 2pE_{ij} - \frac{2}{3} P_{kl}E_{kl}\delta_{ij} + P_{ijk,k} \\
&\quad - \frac{4}{15} q_{k,k}\delta_{ij} + \frac{4}{5} q_{(i,j)} \\
L(c^2 Y_i) &= 2\dot{q}_i + \frac{10}{3} q_i E + 2P_{ijk}E_{jk} + 2q_j u_{i,j} + \frac{8}{5} q_j E_{ij} - \frac{5p}{\rho} P_{ij,j} \\
&\quad - \frac{5p}{\rho} p_{,i} - \frac{2}{\rho} P_{ij}P_{jk,k} - \frac{2}{\rho} P_{ij}p_{,j} + P_{2|ij,j} + \frac{1}{3} P_{4|,i} \\
L(Y_{ijk}) &= \dot{P}_{ijk} + P_{ijk}E + 3P_{l(ij}u_{k),l} - \frac{6}{5} P_{lm(i}\delta_{jk)}E_{lm} + \frac{12}{5} q_{(i}E_{jk)} \\
&\quad - \frac{24}{25} q_l E_{l(i}\delta_{jk)} - \frac{3}{\rho} P_{(ij}P_{k)l,l} - \frac{3}{\rho} P_{(ij}p_{,k)} + \frac{6}{5\rho} P_{l(i}\delta_{jk)}P_{lm,m} \\
&\quad + \frac{6}{5\rho} P_{l(i}\delta_{jk)}p_{,l} + P_{ijkl,l} - \frac{6}{35} \delta_{(ij}P_{2|k)l,l} + \frac{3}{7} P_{2|(ij,k)} \\
L(c^4) &= \dot{P}_{4|} + \frac{7}{3} P_{4|}E + 4P_{2|ij}E_{ij} - \frac{8}{\rho} q_i P_{ij,j} - \frac{8}{\rho} q_i p_{,i} + P_{4|i,i} \\
L(c^2 Y_{ij}) &= \dot{P}_{2|ij} + \frac{5}{3} P_{2|ij}E + 2P_{ijkl}E_{kl} + 2P_{2|k(i}u_{j),k} + \frac{8}{7} P_{2|k(i}E_{j)k} \\
&\quad - \frac{22}{21} P_{2|kl}E_{kl}\delta_{ij} + \frac{14}{15} P_{4|}E_{ij} - \frac{2}{\rho} P_{ijk}P_{kl,l} - \frac{2}{\rho} P_{ijk}p_{,k} \\
&\quad - \frac{28}{5\rho} q_{(i}P_{j)k,k} - \frac{28}{5\rho} q_{(i}p_{,j)} + \frac{28}{15\rho} q_k P_{kl,l}\delta_{ij} + \frac{28}{15\rho} q_k p_{,k}\delta_{ij} \\
&\quad + P_{2|ijk,k} - \frac{2}{15} P_{4|k,k}\delta_{ij} + \frac{2}{5} P_{4|(i,j)} \\
L(c^4 Y_i) &= \dot{P}_{4|i} + \frac{7}{3} P_{4|i}E + P_{4|j}u_{i,j} + 4P_{2|ijk}E_{jk} + \frac{8}{5} P_{4|j}E_{ji} \\
&\quad - \frac{7}{3\rho} P_{4|}P_{ij,j} - \frac{7}{3\rho} P_{4|}p_{,i} - \frac{4}{\rho} P_{2|ij}P_{jk,k} - \frac{4}{\rho} P_{2|ij}p_{,j} \\
&\quad + P_{4|ij,j} + \frac{1}{3} P_{6|,i}
\end{aligned}
\right\}
\tag{25}
$$

and so on, ad infinitum. The new symbol E_{ij} is the divergenceless rate-of-deformation tensor or deviatory rate of deformation defined by

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) - \frac{1}{3} E\delta_{ij} \tag{26}$$

A sample calculation given in appendix B illustrates the method of finding the expressions in equations (25). Note the presence in these expressions of many terms involving permutation of subscripts, such as, for example, the last term in $L\left(Y_{ijk}\right)$ which is

$$\frac{3}{7} P_{2|(ij,k)} = \frac{1}{7}\left(P_{2|ij,k} + P_{2|ki,j} + P_{2|jk,i}\right)$$

and the term in $L\left(Y_{ij}\right)$ which is

$$2P_{k(i}u_{j),k} = \frac{1}{2!}\left(2P_{ki}u_{j,k} + 2P_{kj}u_{i,k}\right)$$

The calculation of $C\left(Y_{2r|s}\right)$ depends on the evaluation of certain integrals whose values are functions of the molecular model chosen. Ikenberry and Truesdell discuss the problem for arbitrary models and set up a systematic procedure for Maxwell molecules which repel each other as the inverse fifth power of the distance. Now a Maxwell gas is an ideal monatomic gas with the equation of state

$$p = R\rho T = nkT \tag{27}$$

where R, the specific gas constant, is equal to the Boltzmann constant k divided by the mass m of a molecule. For this gas the specific heats at constant pressure and constant volume are constants given by $c_v = \frac{3}{2} R$ and $c_p = \frac{5}{2} R$ so that

$$\gamma = \frac{c_p}{c_v} = \frac{5}{3} \tag{28}$$

The specific internal energy is

$$e = c_v T = \frac{3}{2} RT = \frac{3}{2} \frac{p}{\rho} \tag{29}$$

while the specific entropy is

$$\eta = R \frac{3}{2} \log_e \frac{p}{\rho^{5/3}} = R\beta \tag{30}$$

where

$$\beta = \frac{3}{2} \log_e \frac{p}{\rho^{5/3}} \tag{31}$$

is then a dimensionless form of the entropy. Both the coefficient of heat conduction λ and the coefficient of viscosity μ are then proportional to T and these relations will be written

$$\left.\begin{aligned} \lambda &= \alpha\mu \\ \mu &= KT \end{aligned}\right. \tag{32}$$

with

$$\alpha = \frac{15}{4} R \tag{33}$$

Consequently, the Prandtl number is constant and is equal to 2/3 since

$$Pr = \frac{\mu c_p}{\lambda} = \frac{c_p}{\alpha} = \frac{5R}{2\alpha} = \frac{2}{3} \tag{34}$$

Such a gas is a very close approximation to actual monatomic gases at room temperature and above until internal degrees of freedom are excited or the gas becomes ionized.

For the Maxwell gas Ikenberry and Truesdell evaluated the integrals and arrived at certain constants B_1, B_2, and B_4 which depend upon the constant G in the intermolecular force law $f = Gr^{-5}$. These have the following values:

$$\left.\begin{aligned} B_1 &= \sqrt{\frac{G}{2m}}(2.6511) = 2.6511\sqrt{\frac{G}{2m}} \\ B_2 &= 3\sqrt{\frac{G}{2m}}(1.3703) = 4.1109\sqrt{\frac{G}{2m}} \\ B_4 &= 2\sqrt{\frac{G}{2m}}(4.9087) = 9.8174\sqrt{\frac{G}{2m}} \end{aligned}\right\} \tag{35}$$

with $nB_2 = p/\mu$. Because of this last result and the fact that the collision integrals are homogeneous functions of the constants B, it is possible to write the results in the following form:

$$
\left.
\begin{aligned}
mC(1) &= 0 \\
mC\left(Y_i\right) &= 0 \\
mC\left(c^2\right) &= 0 \\
mC\left(Y_{ij}\right) &= -\frac{p}{\mu} P_{ij} \\
mC\left(c^2 Y_i\right) &= -\frac{4p}{3\mu} q_i \\
mC\left(Y_{ijk}\right) &= -\frac{3p}{2\mu} P_{ijk} \\
mC\left(c^4\right) &= -\frac{2p}{3\rho\mu}\left(\rho P_{4|} - 15p^2 + P_{ij}P_{ij}\right) \\
mC\left(c^2 Y_{ij}\right) &= -\frac{7p}{6\rho\mu}\left(\rho P_{2|ij} - pP_{ij} + \frac{4}{7}\left[P_{ik}P_{kj} - \frac{1}{3} P_{kl}P_{kl}\delta_{ij}\right]\right) \\
mC\left(Y_{ijkl}\right) &= -\frac{1}{4}\frac{p}{\mu}\left(6 + \frac{B_4}{B_2}\right)P_{ijkl} + \frac{3}{4\rho}\frac{p}{\mu}\left(2 - \frac{B_4}{B_2}\right)\Bigg[P_{(ij}P_{kl)} - \frac{4}{7} P_{m(i}\delta_{jk}P_{l)m} \\
&\qquad + \frac{2}{35} P_{mn}P_{mn}\delta_{(ij}\delta_{kl)}\Bigg] \\
mC\left(c^4 Y_i\right) &= -\frac{p}{\rho\mu}\left(\rho P_{4|i} - \frac{28}{3} pq_i + \frac{2}{3} P_{ijk}P_{jk} + \frac{28}{15} q_j P_{ij}\right) \\
mC\left(c^2 Y_{ijk}\right) &= -\frac{p}{14\mu}\left(19 + 2\frac{B_4}{B_2}\right)P_{2|ijk} - \frac{9}{7\rho}\frac{p}{\mu}\left(1 - \frac{B_4}{B_2}\right)pP_{ijk} \\
&\qquad - \frac{3}{14\rho}\frac{p}{\mu}\left(8 - \frac{B_4}{B_2}\right)\left(P_{l(i}P_{jk)l} - \frac{2}{5} P_{lm}P_{lm(i}\delta_{jk)}\right) \\
&\qquad + \frac{9p}{35\rho\mu}\left(13 - 6\frac{B_4}{B_2}\right)\left(q_{(i}P_{jk)} - \frac{2}{5} q_l P_{l(i}\delta_{jk)}\right)
\end{aligned}
\right\}
\tag{36}
$$

All but one of the above equations agree with the expressions in reference 6, and the one which does differ is worked out in appendix B as an example of the calculation.

Setting the expressions in equations (25) equal to their corresponding integrals in equations (36) leads to nine exact equations of transfer; if the calculations were continued, the number of such equations would increase indefinitely. The first three equations obtained are the well-known equations of continuity, momentum, and energy while the fourth, fifth, and sixth equations have been obtained before, notably by Grad. However, no one has attempted to use the full set of exact equations. Instead some scheme of truncation like Grad's or a form of iteration is used. The latter leads to sets of approximation equations like those of Burnett. Ikenberry and Truesdell suggest a simple scheme of iteration, which they call Maxwellian iteration, as well as variations of this scheme. Only the Maxwellian iteration method will be considered in detail in this paper.

MAXWELLIAN AND OTHER METHODS OF ITERATION

Maxwellian Iteration

Maxwellian iteration is discussed throughout chapter 1 of reference 6 and what it amounts to in the final formalization is as follows. Replace the exact equation of transfer (11) by the equation

$$mC^{(n+1)}(Q) = L^{(n)}(Q) \tag{37}$$

which indicates that the $n + 1$st iteration for mC is obtained by substituting the nth iteration into L. In order to get started some initiation scheme must be proposed and this must give the Navier-Stokes equations to begin with. Maxwellian iteration, as defined by Ikenberry and Truesdell, then starts with the agreement that to the zeroth approximation

$$P^{(0)}_{2r|s} = \begin{cases} 0 & \text{if} \quad s \neq 0 \\ (2r + 1)!!\rho\left(\frac{p}{\rho}\right)^r & \text{if} \quad s = 0 \end{cases} \tag{38}$$

and takes the functions Q in equation (37) to be $Y_{2r|s}$ in order of increasing $2r + s$ and, for fixed $2r + s$, in order of increasing s. Thus most of the zeroth iterations vanish; for example,

$$P^{(0)}_{ij} = 0$$

$$q^{(0)}_{i} = P^{(0)}_{2|i} = 0$$

$$P^{(0)}_{ijk} = 0$$

The only nonzero iterations which occur in equations (25) are

$$P^{(0)}_{2|} = 3p$$

$$P^{(0)}_{4|} = 15\,\frac{p^2}{\rho}$$

$$P^{(0)}_{6|} = 105\,\frac{p^3}{\rho^2}$$

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since $(2r + 1)!! = (2r + 1)(2r - 1)(2r - 3)\ .\ .\ .\ (3)(1)$.

Except for the first three expressions in equations (25), which are satisfied at all orders of iteration, the expressions to the zeroth order are

$$L^{(0)}\left(Y_{ij}\right) = 2pE_{ij}$$

$$L^{(0)}\left(c^2Y_i\right) = -\frac{5p}{\rho}\,p_{,i} + 5\left(\frac{p^2}{\rho}\right)_{,i} = \frac{4}{3}\,\alpha pT_{,i}$$

$\left(\text{since equation (27) gives } \frac{p_{,i}}{p} = \frac{\rho_{,i}}{\rho} + \frac{T_{,i}}{T} \text{ and } \frac{p}{\rho T} = R = \frac{4\alpha}{15}\right)$

$$L^{(0)}\left(Y_{ijk}\right) = 0$$

(In fact, according to Ikenberry and Truesdell, all $L^{(0)}\left(Y_{2r|s}\right) = 0$ if $s \geqq 3$ so it follows that

$$L^{(0)}\left(Y_{ijkl}\right) = 0$$

$$L^{(0)}\left(c^2Y_{ijk}\right) = 0$$

although these last two were not included in equations (25).)

$$L^{(0)}\left(c^4\right) = 15\left(\frac{p^2}{\rho}\right)^{\cdot} + 35\,\frac{p^2}{\rho}\,E = 20\,\frac{p^2}{\rho}\,\dot{\beta}$$

$\left(\text{since from equation (31) } \dot{\beta} = \frac{3}{2}\frac{\dot{p}}{p} - \frac{5}{2}\frac{\dot{\rho}}{\rho} \text{ while continuity gives } E = -\frac{\dot{\rho}}{\rho}\right)$

$$L^{(0)}\left(c^2Y_{ij}\right) = 14\,\frac{p^2}{\rho}\,E_{ij}$$

$$L^{(0)}\left(c^4Y_i\right) = -35\,\frac{p^2}{\rho^2}\,p_{,i} + 35\left(\frac{p^3}{\rho^2}\right)_{,i} = 70\,\frac{p^3}{\rho^2 T}\,T_{,i}$$

(39)

When one substitutes these expressions into equation (37) with $\eta = 0$ and uses the values of $mC\left(Y_{2r|s}\right)$ given in equations (36) the following set of equations for the first iterations results:

$$-\frac{p}{\mu}\,P_{ij}^{(1)} = 2pE_{ij}$$

$$-\frac{4}{3}\,\frac{p}{\mu}\,q_i^{(1)} = \frac{4}{3}\,\alpha pT_{,i}$$

$$P_{ijk}^{(1)} = 0$$

$$-\frac{2p}{3\rho\mu}\left(\rho P_{4|}^{(1)} - 15p^2 + P_{ij}^{(1)}P_{ij}^{(1)}\right) = 20\,\frac{p^2}{\rho}\,\beta$$

$$-\frac{7p}{6\rho\mu}\left[\rho P_{2|ij}^{(1)} - pP_{ij}^{(1)} + \frac{4}{7}\left(P_{ik}^{(1)}P_{kj}^{(1)} - \frac{1}{3}\,P_{kl}^{(1)}P_{kl}^{(1)}\delta_{ij}\right)\right] = 14\,\frac{p^2}{\rho}\,E_{ij}$$

$$-\frac{p}{4\mu}\left(6 + \frac{B_4}{B_2}\right)P_{ijkl}^{(1)} + \frac{3p}{4\rho\mu}\left(2 - \frac{B_4}{B_2}\right)\left[P_{(ij}^{(1)}P_{kl)}^{(1)} - \frac{4}{7}\,P_{m(i}^{(1)}\delta_{jk}P_{l)m}^{(1)}\right.$$

$$\left. + \frac{2}{35}\,P_{mn}^{(1)}P_{mn}^{(1)}\delta_{(ij}\delta_{kl)}\right] = 0$$

$$-\frac{p}{\rho\mu}\left(\rho P_{4|i}^{(1)} - \frac{28}{3}\,pq_i^{(1)} + \frac{2}{3}\,P_{ijk}^{(1)}P_{jk}^{(1)} + \frac{28}{15}\,q_j^{(1)}P_{ij}^{(1)}\right) = \frac{56}{3}\,\frac{\alpha p^2}{\rho}\,T_{,i}$$

$$-\frac{1}{2}\left(19 + 2\,\frac{B_4}{B_2}\right)P_{2|ijk}^{(1)} - \frac{9p}{\rho}\left(1 - \frac{B_4}{B_2}\right)P_{ijk}^{(1)} - \frac{3}{2\rho}\left(8 - \frac{B_4}{B_2}\right)\left(P_{l(i}^{(1)}P_{jk)l}^{(1)}\right.$$

$$\left. - \frac{2}{5}\,P_{lm}^{(1)}P_{lm(i}^{(1)}\delta_{jk)}\right) + \frac{9}{5\rho}\left(13 - 6\,\frac{B_4}{B_2}\right)\left(q_{(i}^{(1)}P_{jk)}^{(1)} - \frac{2}{5}\,q_l^{(1)}P_{l(i}^{(1)}\delta_{jk)}\right) = 0$$

These algebraic equations are easily solved with the result that the first-order iterations are:

$$P_{ij}^{(1)} = -2\mu E_{ij} \tag{40}$$

$$q_i^{(1)} = \frac{1}{2}\,P_{2|i}^{(1)} = -\alpha\mu T_{,i} \tag{41}$$

$$P^{(1)}_{ijk} = 0 \tag{42}$$

$$P^{(1)}_{4|} = \frac{1}{\rho}\left(15p^2 - 4\mu^2 E_{ij}E_{ij} - 30\mu p\dot{\beta}\right) \tag{43}$$

$$P^{(1)}_{2|ij} = -14\,\frac{\mu p}{\rho}\,E_{ij} - \frac{16\mu^2}{7\rho}\left(E_{ik}E_{kj} - \frac{1}{3}\,E_{kl}E_{kl}\delta_{ij}\right) \tag{44}$$

$$P^{(1)}_{ijkl} = \frac{12\mu^2}{\rho}\left(\frac{2B_2 - B_4}{6B_2 + B_4}\right)\left(E_{(ij}E_{kl)} - \frac{4}{7}\,E_{m(i}\delta_{jk}E_{l)m} + \frac{2}{35}\,E_{mn}E_{mn}\delta_{(ij}\delta_{kl)}\right) \tag{45}$$

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$$P^{(1)}_{4|i} = -28\,\frac{\alpha\mu p}{\rho}\,T_{,i} - \frac{56\alpha\mu^2}{15\rho}\,E_{ij}T_{,j} \tag{46}$$

$$P^{(1)}_{2|ijk} = \frac{36\alpha\mu^2}{5\rho}\left(\frac{13B_2 - 6B_4}{19B_2 + 2B_4}\right)\left(E_{(ij}T_{,k)} - \frac{2}{5}\,T_{,l}E_{l(i}\delta_{jk)}\right) \tag{47}$$

The first six of these equations are identical with the values given in reference 6 but the last two have slightly different numerical coefficients because of the use of the correct value for $mC\left(c^4Y_i\right)$ and an apparent typographic error in the last iteration.

The first two of the iterations (eqs. (40) and (41)) when substituted into the first three of equations (25)

$$\left.\begin{aligned}
&\dot{\rho} + \rho E = 0\\
&\rho\dot{u}_i + p_{,i} + P_{ij,j} = 0\\
&3\dot{p} + 5pE + 2E_{ij}P_{ij} + 2q_{i,i} = 0
\end{aligned}\right\} \tag{48}$$

which are the equations of conservation, give the Navier-Stokes and energy equations for an ideal monatomic gas

$$\left.\begin{aligned}
&\dot{\rho} + \rho E = 0\\
&\rho\dot{u}_i + p_{,i} = 2\left(\mu E_{ij}\right)_{,j}\\
&3\dot{p} + 5pE - 4\mu E_{ij}E_{ij} - 2\left(\lambda T_{,i}\right)_{,i} = 0
\end{aligned}\right\} \tag{49}$$

In order to find the second iterations, insert expressions (40) to (47) into the remaining of equations (25) in order to obtain $L^{(1)}\left(Y_{2r|s}\right)$, then set these equal to the corresponding terms $mC\left(Y_{2r|s}\right)$ with superscript (2) on all of the moments according to the equation

$$mC^{(2)}\left(Y_{2r|s}\right) = L^{(1)}\left(Y_{2r|s}\right)$$

The procedure is straightforward but the calculation is long, so the details will be omitted here. The results are:

$$P_{ij}^{(2)} = -2\mu E_{ij} + \frac{2\mu}{p}\left(\mu E_{ij}\right)^{\cdot} + \frac{2\mu^2}{p} E E_{ij} + \frac{4\mu^2}{p} E_{k(i}u_{j),k} - \frac{4\mu^2}{3p} E_{kl}E_{kl}\delta_{ij} + \frac{4\alpha\mu}{15p}\left[3\left(\mu T_{,(i}\right)_{,j)} - \left(\mu T_{,k}\right)_{,k}\delta_{ij}\right] \tag{50}$$

$$q_i^{(2)} = -\alpha\mu T_{,i} + \frac{3\alpha\mu}{2p}\left(\mu T_{,i}\right)^{\cdot} + \frac{\alpha\mu^2}{2p}\left(5ET_{,i} + 3u_{i,j}T_{,j} + 8E_{ij}T_{,j}\right) + \frac{3\mu}{\rho}\left(\mu E_{ij}\right)_{,j} - \frac{3\mu^2}{p\rho} E_{ij}p_{,j} + \frac{15\mu}{2\rho T}\left(\mu T\dot{\beta}\right)_{,i} + \frac{12\mu}{7p}\left(\frac{\mu^2}{\rho} E_{ij}E_{jk}\right)_{,k} + \frac{3\mu}{7p}\left(\frac{\mu^2}{\rho} E_{jk}E_{jk}\right)_{,i} + \frac{6\mu^2}{p\rho} E_{ij}\left(\mu E_{jk}\right)_{,k} \tag{51}$$

$$\begin{aligned} P_{ijk}^{(2)} = {} & \frac{6\mu^2}{\rho T} T_{,(i}E_{jk)} - \frac{12\mu^2}{5\rho T} T_{,l}E_{l(i}\delta_{jk)} + \frac{8\mu^2}{p\rho} E_{(ij}\left(\mu E_{k)l}\right)_{,l} \\ & - \frac{16\mu^2}{5p\rho} E_{l(i}\delta_{jk)}\left(\mu E_{lm}\right)_{,m} - \frac{8\mu}{5}\left(\frac{\mu}{\rho}\delta_{(ij}E_{k)l}\right)_{,l} + 4\mu\left(\frac{\mu}{\rho}E_{(ij}\right)_{,k)} \\ & - \frac{64\mu}{245p}\left[\frac{\mu^2}{\rho}\delta_{(ij}\left(E_{k)l}E_{lm} - \frac{1}{3}\delta_{k)m}E_{rs}E_{rs}\right)\right]_{,m} + \frac{32\mu}{49p}\left[\frac{\mu^2}{\rho}\left(E_{l(i}E_{lj}\right.\right. \\ & \left.\left.- \frac{1}{3}E_{mn}E_{mn}\delta_{(ij}\right)\right]_{,k)} - \frac{8\mu}{p}\left(\frac{2B_2 - B_4}{6B_2 + B_4}\right)\left[\frac{\mu^2}{\rho}\left(E_{(ij}E_{kl)}\right.\right. \\ & \left.\left.- \frac{4}{7}E_{m(i}\delta_{jk}E_{l)m} + \frac{2}{35}E_{mn}E_{mn}\delta_{(ij}\delta_{kl)}\right)\right]_{,l} \end{aligned} \tag{52}$$

These iterations are the same as those given in reference 6 but the next two are somewhat different and therefore correct the errors in equations (11.4) and (11.5) of reference 6. They are:

$$
\begin{aligned}
P^{(2)}_{4|} = {} & \frac{15p^2}{\rho} - \frac{1}{\rho} P^{(2)}_{ij} P^{(2)}_{ij} + \frac{6\mu}{p}\left(\frac{\mu^2}{\rho} E_{ij} E_{ij}\right)^{\cdot} + 14 \frac{\mu^3}{p\rho} E E_{ij} E_{ij} \\
& - 30 \frac{p\mu}{\rho} \dot{\beta} + 45 \frac{\mu}{\rho T}(\mu T \dot{\beta})^{\cdot} + 105 \frac{\mu^2}{\rho} E\dot{\beta} + 84 \frac{\mu^2}{\rho} E_{ij} E_{ij} \\
& + \frac{96}{7} \frac{\mu^3}{p\rho} E_{ij} E_{jk} E_{ki} - 45 \frac{\mu^2}{\rho^2 T} T_{,i} p_{,i} + \frac{315\mu p}{2\rho^2 T^2}(\mu T T_{,i})_{,i} \\
& + 111 \frac{\mu^2}{\rho^2 T}\left(\mu E_{ij}\right)_{,i} T_{,j} + 21 \frac{\mu^2}{\rho T} E_{ij}\left(\frac{\mu}{\rho} T_{,j}\right)_{,i} \qquad (53)
\end{aligned}
$$

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$$
\begin{aligned}
P^{(2)}_{2|ij} = {} & -14 \frac{\mu p}{\rho} E_{ij} + 14 \frac{\mu}{\rho}\left(\mu E_{ij}\right)^{\cdot} + 32 \frac{\mu^2}{\rho} E_{ij}\dot{\beta} + 14 \frac{\mu^2}{\rho} E E_{ij} \\
& + 28 \frac{\mu^2}{\rho} E_{k(i} u_{j),k} + \frac{96}{7} \frac{\mu^2}{\rho} E_{ik} E_{kj} - \frac{292}{21} \frac{\mu^2}{\rho} E_{kl} E_{kl} \delta_{ij} \\
& - \frac{4}{21\rho}\left(3P^{(2)}_{ik} P^{(2)}_{kj} - P^{(2)}_{kl} P^{(2)}_{kl} \delta_{ij}\right) - 18 \frac{\mu^2}{\rho^2 T} T_{,(i} p_{,j)} + 6 \frac{\mu^2}{\rho^2 T} T_{,k} p_{,k} \delta_{ij} \\
& + 36 \frac{\mu^2 p}{\rho^2 T^2} T_{,i} T_{,j} - 12 \frac{\mu^2 p}{\rho^2 T^2} T_{,k} T_{,k} \delta_{ij} + 39 \frac{\mu p}{\rho^2 T}\left(\mu T_{,(i}\right)_{,j)} \\
& - 13 \frac{\mu p}{\rho^2 T}\left(\mu T_{,k}\right)_{,k} \delta_{ij} + \frac{32}{49} \frac{\mu}{p\rho}\left[\mu^2\left(3E_{ki} E_{jk} - E_{kl} E_{kl} \delta_{ij}\right)\right]^{\cdot} \\
& + \frac{256}{147} \frac{\mu^3}{p\rho} E\left(3E_{ik} E_{kj} - E_{kl} E_{kl} \delta_{ij}\right) + \frac{64}{49} \frac{\mu^3}{p\rho}\left(3E_{kl} E_{l(i} - E_{mn} E_{mn} \delta_{k(i}\right) u_{j),k} \\
& + \frac{16}{1715} \frac{\mu^3}{p\rho}\left(240 E_{ik} E_{jl} E_{kl} + 263 E_{kl} E_{kl} E_{ij} - 220 E_{kl} E_{lm} E_{mk} \delta_{ij}\right) \\
& + \frac{8\mu}{5\rho T}\left[3\left(\frac{\mu^2}{\rho} T_{,k} E_{k(i}\right)_{,j)} - \left(\frac{\mu^2}{\rho} T_{,k} E_{kl}\right)_{,l} \delta_{ij}\right] + \frac{12\mu^2}{\rho^2 T}\left[3T_{,(i}\left(\mu E_{j)k}\right)_{,k}\right. \\
& \left. - T_{,k}\left(\mu E_{kl}\right)_{,l} \delta_{ij}\right] - \frac{12}{7} \frac{\mu}{p} P^{(1)}_{ijkl} E_{kl} - \frac{6\mu}{7p} P^{(1)}_{2|ijk,k} \qquad (54)
\end{aligned}
$$

The other second iterations $P^{(2)}_{ijkl}$, $P^{(2)}_{4|i}$, and $P^{(2)}_{2|ijk}$ were not found because they are not needed to obtain the third iterations for P_{ij} and q_i, and, because of the great length of the calculation, it was carried just far enough to obtain $P^{(3)}_{ij}$ and $q^{(3)}_i$ whose values turn out to be, in agreement with equations (12.1) and (12.2) of reference 6,

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$$P^{(3)}_{ij} = -\frac{\mu}{p}\Big[\dot{P}^{(2)}_{ij} + EP^{(2)}_{ij} + 2P^{(2)}_{k(i}u_{j),k} + 2pE_{ij} - \frac{2}{3}P^{(2)}_{kl}E_{kl}\delta_{ij} + P^{(2)}_{ijk,k} - \frac{4}{15}q^{(2)}_{k,k}\delta_{ij} + \frac{4}{5}q^{(2)}_{(i,j)}\Big] \tag{55}$$

$$q^{(3)}_i = -\frac{3\mu}{4p}\Big[2\dot{q}^{(2)}_i + \frac{10}{3}Eq^{(2)}_i + 2P^{(2)}_{ijk}E_{jk} + 2q^{(2)}_j u_{i,j} + \frac{8}{5}q^{(2)}_j E_{ji} - \frac{5p}{\rho}P^{(2)}_{ij,j} - \frac{5p}{\rho}p_{,i} - \frac{2}{\rho}P^{(2)}_{ij}P^{(2)}_{jk,k} - \frac{2}{\rho}P^{(2)}_{ij}p_{,j} + P^{(2)}_{2|ij,j} + \frac{1}{3}P^{(2)}_{4|,i}\Big] \tag{56}$$

Upon substitution for $P^{(2)}_{ij}$, $q^{(2)}_i$, $P^{(2)}_{ijk}$, $P^{(2)}_{2|ij}$, and $P^{(2)}_{4|}$ from equations (50) to (54), the third iterations for P_{ij} and q_i will now be had in terms of the quantities of state p, ρ, T, and u_i. Using these third iterations in equation (48) will then give a system of five partial differential equations which are similar to the Burnett equations.

This is as far as the method of Maxwellian iteration is carried by Ikenberry and Truesdell and as far as it will be carried here. Neither the initiation agreement (38) nor the method of iteration defined by equation (37) is the only method possible, and it could be that a better approach is possible.

Other Methods of Iteration

Ikenberry and Truesdell also propose the following methods of iteration:

(1) Atemporal Maxwellian iteration. This method is closest to the Chapman-Enskog process and gives the Burnett terms but is rejected by the authors of reference 6 because it adds unnecessary complications to the simpler method of Maxwellian iteration.

(2) Grad's method. This is Grad's method of truncation of the exact equations found above; by setting $L(Y_{2r|s}) = mC(Y_{2r|s})$ the equations of Grad can be found for Maxwell molecules. Thus if

$$p_{ijkl} = 3RT\left(2p_{(ij}\delta_{kl)} - p\delta_{(ij}\delta_{kl)}\right)$$

Grad's 20-moment approximation is obtained, and if, in addition,

$$p_{ijk} = 0$$

the Grad 13-moment approximation is found.

(3) Integral iteration. This is a modification of iteration equation (37) which requires an integration to be performed at each step. It also requires a knowledge of initial values of all the moments used. Even when linearized, the resulting equations remain too complex for actual solution so some scheme of approximation is still necessary. The method does not seem to be any better than the Maxwellian iteration, although it does give a correct solution to the problem of time-dependent shear flow. In a personal communication, Trusedell expressed the belief that integral iteration is better than Maxwellian iteration in that it probably leads to the general solution of the initial-value problem. However, aerodynamicists are more concerned with the steady-state problem.

(4) Scheme using maximum available information. This is a variation of Maxwellian iteration where at each step the highest known iteration is used. It has the advantage of introducing higher-order terms faster than Maxwellian iteration introduces them.

(5) Iteration by powers. This is similar to the Hilbert-Enskog expansion in powers of a parameter which is finally set equal to 1. The resulting equations have fewer terms than are necessary in Maxwellian iteration.

(6) Truesdell iteration. This is suggested as the name for what is called in reference 6 "a scheme which treats algebraically like terms on an equal footing." Instead of obtaining iterations which are essentially expressed in increasing powers of μ, such as in the other schemes including Maxwellian iteration, this treats all of the terms in each equation of transfer alike. Only the higher moments and the time derivatives are replaced by approximations. The result is that each iteration is a rational function instead of a polynomial in μ. This scheme shows considerably more promise than any of the others and one attempt to apply it to plane Couette flow is given in reference 8. Trusedell finds that it is also best for the case of time-dependent shear flow.

(7) Iteration in equations of moments. This is a variation of Maxwellian iteration which has a different initiation agreement. It is not developed or used in reference 6 and does not appear to be any better than the Maxwellian iteration.

(8) Iteration from free-molecular flow. This method starts with the initiation agreement that the zeroth iterations are the values of the moments of free-molecular flow. However, it is not developed, and elsewhere in his paper Truesdell effectively shows that the whole kinetic theory as based upon moments is certainly not appropriate for free-molecular flows.

It would appear that the method of Maxwellian iteration is probably the easiest method to use in theoretical aerodynamics if a solution of equations (48) is wanted. Actually, the exact equations of transfer are equivalent to the Boltzman equation; a nonlinear integrodifferential equation is thus replaced by an infinite system of first-order nonlinear partial differential equations and for problems in continuum or near-continuum flow the latter are certainly more appropriate. The question of boundary values for an infinite set of moments has not been and probably cannot be settled because of the lack of physical significance of the higher moments. Thus it is necessary in any physical problem to resort to some sort of iteration so that these higher moments can be expressed in terms of the physical variables of state and their derivatives. Even then the question of boundary conditions on these variables has not been fully answered since a proper formulation of boundary conditions seems to depend upon a knowledge of the distribution function $F(\vec{x},\vec{\xi},t)$ in the neighborhood of the wall where the boundary condition is taken. To specify F would not be in the spirit of the above theory, which is independent of F.

There is one problem of great physical interest for which boundary conditions are not required and this is that of the structure of a steady, plane shock wave. Even though this problem leads to one of the simplest forms of the equations, the equations are still very complex. Still an approximate solution can be found, and it will be presented in the next section of this paper.

STRUCTURE OF PLANE SHOCK WAVE

Derivation of Equations for Plane Shock Wave

The macroscopic differential equations (48) for a Maxwell gas with the pressure tensor and heat-flux vector given by the third Maxwellian iteration (eqs. (55) and (56)) can be used to calculate the thickness of a plane, steady, shock wave as suggested by Ikenberry and Truesdell (footnote p. 41, ref. 6). Their suggestion was to go only as far as the second iterations for P_{ij} and q_i. Talbot and Sherman (ref. 9, p. 16) refer to a series solution they found for the extended Burnett equations, although the results they give are those of a numerical integration. The numerical integration is so much better that considerable doubt is thrown upon any series solution. Since only the series solution of the Ikenberry-Truesdell equations was found, however, the method used and results obtained are reported upon here.

Since the original series method of Talbot and Sherman included
linear terms from the third-order Burnett approximation, it was con- W
sidered desirable to go further and include all terms in μ^3 from the 1
third Maxwellian iteration. These turned out to be quite numerous 3
and added some complexity to the calculations but still the labor was 4
not so great as it would have been if all of the third-iteration terms had been included. Both the author of this paper and one of his students, Mr. Richard A. Gregory, carried through the derivations and calculations more or less independently until the same results were obtained. Thus the results presented here are considered to be correct within the limitations of the series method.

In the one-dimensional steady flow of a gas there is only one space variable x_1 which is denoted by x and only one component of the velocity vector u_1 denoted by u. The hydrodynamic derivative reduces to

$$(\dot{\ }) \equiv \frac{D(\ )}{Dt} = u\,\frac{d(\ )}{dx} = u(\ )'$$

so that the equations of conservation (48) become

$$\left.\begin{aligned} &u\rho' + \rho E = 0 \\ &\rho u u' + p' + P_{xx}' = 0 \\ &P_{xy}' = P_{xz}' = 0 \\ &3up' + 5pE + 2P_{ij}E_{ij} + 2q_x' = 0 \end{aligned}\right\} \qquad (57)$$

The prime denotes differentiation with respect to x. Since

$$E \equiv u_{i,i} = u'$$

$$E_{xx} = u' - \frac{1}{3}E = \frac{2}{3}u'$$

$$E_{yy} = E_{zz} = -\frac{1}{3} E = -\frac{1}{3} u'$$

$$E_{xy} = E_{xz} = E_{yz} = 0$$

where subscripts x,y,z denote the subscripts 1, 2, 3, respectively, of the tensors P_{ij} and E_{ij}, equations (57) reduce to

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$$\left.\begin{aligned} &\frac{d}{dx}(\rho u) = 0 \\ &\frac{d}{dx}\left(\rho u^2 + p + P_{xx}\right) = 0 \\ &3up' + 5pu' + 2\left[\frac{2}{3} u'P_{xx} - \frac{1}{3} u'\left(P_{yy} + P_{zz}\right)\right] + 2q_x' = 0 \end{aligned}\right\} \tag{58}$$

The first two equations have the integrals

$$\rho u = m = \text{Constant} \tag{59}$$

$$\rho u^2 + p + P_{xx} = P = \text{Constant} \tag{60}$$

and since $P_{xx} + P_{yy} + P_{zz} = 0$, the third equation can be written

$$\frac{3}{2}(up)' + u'\left(p + P_{xx}\right) + q_x' = 0$$

Using equation (60), this becomes

$$\frac{d}{dx}\left(\frac{3}{2} up + uP - \frac{1}{2} mu^2 + q_x\right) = 0$$

with the integral

$$\frac{3}{2} up + uP - \frac{1}{2} \rho u^3 + q_x = \frac{1}{2} Q = \text{Constant}$$

Eliminating P between this equation and equation (60) gives the energy equation

$$5up + \rho u^3 + 2uP_{xx} + 2q_x = Q \tag{61}$$

Since $p = R\rho T$ the three equations (59) to (61) reduce to two:

$$\left.\begin{aligned} mu + \frac{mRT}{u} + P_{xx} &= P \\ 5mRT + mu^2 + 2uP_{xx} + 2q_x &= Q \end{aligned}\right\} \tag{62}$$

which indicate explicitly that the problem involves only two unknown functions u and T of x which have to be found. First the appropriate iteration for P_{xx} and q_x has to be found and substituted into equation (62).

The first iterations given by equations (40) and (41) are

$$P_{xx}^{(1)} = -\frac{4}{3}\,\mu u'$$

$$q_x^{(1)} = -\alpha\mu T'$$

Introducing these into equations (62) gives the Navier-Stokes equations for the simple one-dimensional flow. The second iterations are given by equations (50) and (51) which, when they are written for one-dimensional flow and like terms are combined, will give

$$P_{xx}^{(2)} = -\frac{4}{3}\,\mu u' + \frac{4\mu}{3p}\,u(\mu u')' + \frac{28\mu^2}{9p}\,u'^2 + \frac{2\mu}{\rho T}(\mu T')'$$

$$q_x^{(2)} = -\alpha\mu T' + \frac{3\alpha\mu}{2p}\,u(\mu T')' + \frac{20\alpha\mu^2}{3p}\,u'T' + \frac{2\mu}{\rho}(\mu u')' - \frac{2\mu^2}{p\rho}\,u'p'$$
$$+ \frac{15\mu}{2\rho T}\,(\mu u T\beta')' + \frac{22\mu}{21p}\left[\frac{\mu^2}{\rho}(u')^2\right]' + \frac{8\mu^2}{3p\rho}\,u'(\mu u')'$$

Evaluation of the derivatives in terms of those of just u and T by means of the formulas of appendix C reduces these iterations to

$$P_{xx}^{(2)} = -\frac{4}{3}\,\mu u' + \frac{4\mu^2}{3p}\,uu'' + \frac{28\mu^2}{9p}\,u'^2 + \frac{4\mu^2}{3pT}\,uu'T' + \frac{2\mu^2}{\rho T^2}\,T'^2 + \frac{2\mu^2}{\rho T}\,T'' \tag{63}$$

$$q_x^{(2)} = -\alpha\mu T' + \frac{19\mu^2}{2\rho}\,u'' + \frac{2\mu^2}{\rho u}(u')^2 + \frac{205\mu^2}{4\rho T}\,u'T' + \frac{135\mu^2}{8\rho T^2}\,u(T')^2$$
$$+ \frac{135\mu^2}{8\rho T}\,uT'' + \frac{100\mu^3}{21p\rho}\,u'u'' + \frac{22\mu^3}{21p\rho u}(u')^3 + \frac{100\mu^3}{21p\rho T}(u')^2T' \tag{64}$$

These expressions are quite similar to the corresponding ones in equation (B7) of reference 9, yet they differ significantly, especially the expression for $q_x^{(2)}$. It is seen that Maxwellian iteration is not in powers of μ as the last three terms in equation (64) show. The third iterations will include more terms in μ^3 as well as some terms in μ^4.

As equations (55) and (56) show, the third iterations depend on the second iterations $P_{ij}^{(2)}$, $q_i^{(2)}$, $P_{ijk}^{(2)}$, $P_{2|ij}^{(2)}$, and $P_{4|}^{(2)}$ which in turn depend upon a knowledge of the first iterations $P_{ijkl}^{(1)}$ and $P_{2|ijk}^{(1)}$. Even for this simple case the number of terms in all of these iterations becomes tremendous, yet the calculation is straightforward although time consuming. Since Talbot and Sherman originally included terms linear in μ^3 but not all of the terms in μ^3 and none in μ^4 for the Burnett equations, it was considered sufficient if all terms to μ^3 in the third iterations for P_{xx} and q_x were included but no terms in μ^4. Then in the second iterations above only terms in μ^2 need be retained and the first iterations $P_{ijk}^{(1)}$ and $P_{2|ijk}^{(1)}$ will not be used since they are themselves proportional to μ^2.

The third iteration for this one-dimensional flow according to equations (55) and (56) becomes

$$P_{xx}^{(3)} = -\frac{\mu}{p}\left(u\frac{d}{dx}P_{xx}^{(2)} + 3u'P_{xx}^{(2)} + \frac{4}{3}pu' - \frac{2}{3}E_{kl}P_{kl}^{(2)} + \frac{d}{dx}P_{xxx}^{(2)} + \frac{8}{15}\frac{d}{dx}q_x^{(2)}\right) \tag{65}$$

$$q_x^{(3)} = -\frac{3\mu}{4p}\left(-\frac{5p}{\rho}p' + 2u\frac{d}{dx}q_x^{(2)} + \frac{32}{5}u'q_x^{(2)} + 2P_{xjk}^{(2)}E_{jk} - \frac{5p}{\rho}\frac{d}{dx}P_{xx}^{(2)} - \frac{2p'}{\rho}P_{xx}^{(2)} - \frac{2}{\rho}P_{xx}^{(2)}\frac{d}{dx}P_{xx}^{(2)} + \frac{d}{dx}P_{2|xx}^{(2)} + \frac{1}{3}\frac{d}{dx}P_{4|}^{(2)}\right) \tag{66}$$

In equation (65)

$$E_{kl}P_{kl}^{(2)} = \frac{2}{3}u'P_{xx}^{(2)} - \frac{1}{3}u'\left(P_{yy}^{(2)} + P_{zz}^{(2)}\right) = u'P_{xx}^{(2)}$$

since

$$P_{yy}^{(2)} + P_{zz}^{(2)} = -P_{xx}^{(2)}$$

Also

$$P_{yy}^{(2)} = P_{zz}^{(2)} = -\frac{1}{2} P_{xx}^{(2)} = \frac{2}{3} \mu u' + O\left(\mu^2\right)$$

so that

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$$\left(P_{xx}^{(2)}\right)^2 - \left(P_{yy}^{(2)}\right)^2 = \frac{4}{3} \mu^2 (u')^2 + O\left(\mu^3\right)$$

where $O(\ )$ denotes order of.

From equations (52) to (54), retaining only terms in μ^2,

$$P_{xxx}^{(2)} = \frac{8\mu^2}{5\rho} u'' + \frac{8\mu^2}{5\rho u}(u')^2 + \frac{4\mu^2}{\rho T} u'T' + O\left(\mu^3\right)$$

$$P_{xyy}^{(2)} = P_{xzz}^{(2)} = -\frac{4\mu^2}{5\rho} u'' - \frac{4\mu^2}{5\rho u}(u')^2 - \frac{2\mu^2}{\rho T} u'T' + O\left(\mu^3\right)$$

$$P_{2|xx}^{(2)} = -\frac{28}{3} \frac{\mu p}{\rho} u' + \frac{28\mu^2}{3\rho} uu'' + \frac{2876}{63} \frac{\mu^2}{\rho}(u')^2 + \frac{124\mu^2}{3\rho T} uu'T'$$
$$+ 12 \frac{\mu^2 p}{\rho^2 u T} u'T' + 38 \frac{\mu^2 p}{\rho^2 T^2}(T')^2 + 26 \frac{\mu^2 p}{\rho^2 T} T'' + O\left(\mu^3\right)$$

$$P_{4|}^{(2)} = 15 \frac{p^2}{\rho} - 30 \frac{\mu p}{\rho} u' - 45 \frac{\mu p}{\rho T} uT' + 475 \frac{\mu^2}{3\rho}(u')^2 + 45 \frac{\mu^2}{\rho} uu''$$
$$+ 315 \frac{\mu^2}{\rho T} uu'T' + 45 \frac{\mu^2 p}{\rho^2 u T} u'T' + \frac{135\mu^2}{2\rho T^2} u^2 (T')^2 + 270 \frac{\mu^2 p}{\rho^2 T^2}(T')^2$$
$$+ \frac{135\mu^2}{2\rho T} u^2 T'' + 315 \frac{\mu^2 p}{2\rho^2 T} T'' + O\left(\mu^3\right)$$

Taking derivatives of these iterations and the ones in equations (63) and (64) is long and tedious; when the results are substituted into equations (65) and (66) and like terms are combined, the final forms for $P_{xx}^{(3)}$ and $q_x^{(3)}$ are

$$
\begin{aligned}
P_{xx}^{(3)} = & -\frac{4}{3}\,\mu u' + \frac{28\mu^2}{9p}(u')^2 + \frac{4\mu^2}{3p}\,uu'' + \frac{4\mu^2}{3pT}\,uu'T' + \frac{2\mu^2}{\rho T^2}(T')^2 \\
& + \frac{2\mu^2}{\rho T}\,T'' - \frac{20\mu^3}{3p\rho}\,u''' - \frac{4\mu^3}{3p^2}\,u^2u''' - 12\,\frac{\mu^3}{p^2}\,uu'u'' - \frac{12\mu^3}{p\rho u}\,u'u'' \\
& - \frac{280\mu^3}{27p^2}(u')^3 - \frac{8\mu^3}{3p^2T}\,u^2u''T' - \frac{134\mu^3}{3p\rho T}\,u''T' - \frac{80\mu^3}{9p^2T}\,u(u')^2T' \\
& - \frac{110\mu^3}{3p\rho uT}(u')^2T' - \frac{56\mu^3}{p\rho T^2}\,u'(T')^2 - \frac{4\mu^3}{3p^2T}\,u^2u'T'' - \frac{33\mu^3}{p\rho T^2}\,uT'T'' \\
& - \frac{56\mu^3}{p\rho T}\,u'T'' - \frac{11\mu^3}{p\rho T}\,uT''' + O\left(\mu^4\right)
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
q_x^{(3)} = & -\frac{15\mu p}{4\rho T}\,T' + \frac{19\mu^2}{2\rho}\,u'' + \frac{205\mu^2}{4\rho T}\,u'T' + \frac{135\mu^2}{8\rho T^2}\,u(T')^2 + \frac{135\mu^2}{8\rho T}\,uT'' \\
& + \frac{2\mu^2}{\rho u}(u')^2 - \frac{55\mu^3}{2p\rho}\,uu''' - \frac{6115\mu^3}{28p\rho}\,u'u'' - \frac{2207\mu^3}{28p\rho u}(u')^3 - \frac{1917\mu^3}{8p\rho T}\,uu''T' \\
& - \frac{37801\mu^3}{56p\rho T}(u')^2T' - \frac{81\mu^3}{4\rho^2uT}\,u''T' - \frac{2935\mu^3}{8p\rho T^2}\,uu'(T')^2 - \frac{132\mu^3}{\rho^2uT^2}\,u'(T')^2 \\
& - \frac{93\mu^3}{\rho^2T^3}(T')^3 - \frac{2911\mu^3}{8p\rho T}\,uu'T'' - \frac{2025\mu^3}{16p\rho T^2}\,u^2T'T'' - \frac{597\mu^3}{8\rho^2uT}\,u'T'' \\
& - \frac{1137\mu^3}{4\rho^2T^2}\,T'T'' - \frac{411\mu^3}{8\rho^2T}\,T''' - \frac{675\mu^3}{16p\rho T}\,u^2T''' + O\left(\mu^4\right)
\end{aligned}
\tag{68}
$$

Equations (62) together with equations (67) and (68) are now the equations of motion. By imposing the Rankine-Hugoniot conditions on the flow far upstream and far downstream

$$
\left.
\begin{aligned}
u_i + \frac{RT_i}{u_i} &= u_f + \frac{RT_f}{u_f} = \frac{P}{m} \\
u_i^2 + 5RT_i &= u_f^2 + 5RT_f = \frac{Q}{m}
\end{aligned}
\right\}
\tag{69}
$$

where subscripts i and f refer to upstream and downstream conditions, the problem becomes that of the plane, steady shock wave.

Series Solution of Equations for Plane Shock Wave

Take a series expansion in powers of Grad's expansion parameter ϵ which is defined as

$$\epsilon = \sqrt{25 - 16 \frac{mQ}{P^2}} \tag{70}$$

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In a Maxwell gas the Mach number squared is

$$M^2 = \frac{3\rho u^2}{5p} = \frac{3u^2}{5RT}$$

and the Rankine-Hugoniot conditions (69) become

$$u_i\left(1 + \frac{3}{5M_i{}^2}\right) = u_f\left(1 + \frac{3}{5M_f{}^2}\right) = \frac{P}{m}$$

$$u_i{}^2\left(1 + \frac{3}{M_i{}^2}\right) = u_f{}^2\left(1 + \frac{3}{M_f{}^2}\right) = \frac{Q}{m}$$

Then

$$25 - 16 \frac{mQ}{P^2} = 225\left(\frac{M_i{}^2 - 1}{5M_i{}^2 + 3}\right)^2$$

and

$$\epsilon = \frac{15\left(M_i{}^2 - 1\right)}{5M_i{}^2 + 3} \tag{71}$$

Let t and w be the dimensionless forms of temperature and velocity defined by

$$u = \frac{P}{m}\,\frac{5 + \epsilon w}{8} \tag{72}$$

$$T = \frac{P^2}{m^2 R} \, \frac{15 + 2\epsilon t - \epsilon^2}{64} \tag{73}$$

Then

$$\rho = \frac{m}{u} = \frac{m^2}{P} \, \frac{8}{5 + \epsilon w} \tag{74}$$

$$p = \frac{mRT}{u} = \frac{P}{8} \, \frac{15 + 2\epsilon t - \epsilon^2}{5 + \epsilon w} \tag{75}$$

and the Rankine-Hugoniot boundary conditions of equations (69) reduce to the simple form

$$\left.\begin{array}{l} \left.\begin{array}{l} w_i = 1 \\ t_i = -1 \end{array}\right\} \quad \text{upstream when} \quad x \to -\infty \\ \\ \left.\begin{array}{l} w_f = -1 \\ t_f = 1 \end{array}\right\} \quad \text{downstream when} \quad x \to \infty \end{array}\right\} \tag{76}$$

Other parameters are introduced, one of which is

$$L = \frac{\mu}{m} = \frac{\lambda}{\alpha m} = \frac{4\lambda}{15Rm} \tag{77}$$

Let

$$\left.\begin{array}{l} \dfrac{L}{L_*} = \dfrac{T}{T_*} \\ \\ T_* = \dfrac{2}{\gamma + 1} T_o = \dfrac{3}{4} T_o \end{array}\right\} \tag{78}$$

Here L_* is a reference length corresponding to μ_* at the temperature T_* which occurs when $M = 1$, so that T_o is the stagnation temperature. Let

$$\tau = \frac{m^2}{P^2} RT = \frac{15 + 2\epsilon t - \epsilon^2}{64} \tag{79}$$

so

$$\frac{L}{L_*} = \frac{\tau}{\tau_*} \tag{80}$$

where

$$\tau_* = \frac{15}{64}\left(1 - \frac{\epsilon^2}{25}\right) \tag{81}$$

The dimensionless length, used in place of x, is then

$$\eta = \frac{256}{525}\,\frac{\epsilon\tau_* x}{L_*} \tag{82}$$

and this is related to Grad's reference length

$$y = \frac{4}{35}\,\frac{\epsilon x}{L_*} \tag{83}$$

by

$$\eta = \left(1 - \frac{\epsilon^2}{25}\right)y \tag{84}$$

Equations (62) now become in terms of t and w

$$\left.\begin{aligned} &2\epsilon(w + t) + \epsilon^2\left(w^2 - 1\right) + 8(5 + \epsilon w)\frac{P_{xx}}{P} = 0 \\ &10\epsilon(w + t) + \epsilon^2\left(w^2 - 1\right) + 16(5 + \epsilon w)\frac{P_{xx}}{P} + \frac{128m}{P^2}\,q_x = 0 \end{aligned}\right\} \tag{85}$$

From the definition of η

$$\frac{d}{dx} = \frac{256}{525}\,\frac{\epsilon\tau_*}{L_*}\,\frac{d}{d\eta} \tag{86}$$

so if the prime is now used to denote differentiation with respect to η, the expressions for $P_{xx}^{(3)}$ and $q_x^{(3)}$ given by equations (67) and (68) become

$$\frac{P_{xx}^{(3)}}{P} = -\frac{2\epsilon^2}{3(525)}\left(15 + 2\epsilon t - \epsilon^2\right)w' + \frac{56\epsilon^4}{9(525)^2}(5 + \epsilon w)\left(15 + 2\epsilon t - \epsilon^2\right)(w')^2$$

$$+ \frac{8\epsilon^3}{3(525)^2}(5 + \epsilon w)^2\left(15 + 2\epsilon t - \epsilon^2\right)w'' + \frac{16\epsilon^4}{3(525)^2}(5 + \epsilon w)^2 w't'$$

$$+ \frac{16\epsilon^4}{(525)^2}(5 + \epsilon w)(t')^2 + \frac{8\epsilon^3}{(525)^2}(5 + \epsilon w)\left(15 + 2\epsilon t - \epsilon^2\right)t''$$

$$- \frac{32\epsilon^4}{3(525)^3}(5 + \epsilon w)^2\left(15 + 2\epsilon t - \epsilon^2\right)\left[5\left(15 + 2\epsilon t - \epsilon^2\right) + (5 + \epsilon w)^2\right]w'''$$

$$- \frac{176\epsilon^4}{(525)^3}(5 + \epsilon w)^3\left(15 + 2\epsilon t - \epsilon^2\right)t''' + 0\left(\epsilon^5\right)$$

$$\frac{mq_x^{(3)}}{P^2} = -\frac{\epsilon^2}{1120}\left(15 + 2\epsilon t - \epsilon^2\right)t' + \frac{19\epsilon^3}{8(525)^2}(5 + \epsilon w)\left(15 + 2\epsilon t - \epsilon^2\right)^2 w''$$

$$+ \frac{41\epsilon^4}{840(525)}(5 + \epsilon w)\left(15 + 2\epsilon t - \epsilon^2\right)w't' + \frac{27\epsilon^4}{840(525)}(5 + \epsilon w)^2(t')^2$$

$$+ \frac{27\epsilon^3}{1680(525)}(5 + \epsilon w)^2\left(15 + 2\epsilon t - \epsilon^2\right)t'' + \frac{\epsilon^4}{2(525)^2}\left(15 + 2\epsilon t - \epsilon^2\right)^2(w')^2$$

$$- \frac{11\epsilon^4}{210(525)^2}(5 + \epsilon w)^3\left(15 + 2\epsilon t - \epsilon^2\right)^2 w'''$$

$$- \frac{411\epsilon^4}{4(525)^3}(5 + \epsilon w)^2\left(15 + 2\epsilon t - \epsilon^2\right)^2 t'''$$

$$- \frac{675\epsilon^4}{8(525)^3}(5 + \epsilon w)^4\left(15 + 2\epsilon t - \epsilon^2\right)t''' + 0\left(\epsilon^5\right)$$

When these are substituted into equations (85) and the terms arranged in increasing powers of ϵ only as far as ϵ^4, the differential equations to be solved are

$$
\begin{aligned}
&2\epsilon(t + w) + \epsilon^2\left(w^2 - 1 - \frac{16}{21}\,w'\right) + \frac{16\epsilon^3}{105}\left(-\,\frac{2t + 3w}{3}\,w' + \frac{20}{21}\,w'' + \frac{4}{7}\,t''\right) \\
&+ \frac{64\epsilon^4}{105}\left[\frac{5 - 2tw}{60}\,w' + \frac{9w + 2t}{63}\,w'' + \frac{2}{105}(3w + t)t'' + \frac{1}{9}(w')^2 + \frac{2}{63}\,w't'\right. \\
&\left. + \frac{2}{105}(t')^2 - \frac{80}{(21)^2}\,w''' - \frac{22}{147}\,t'''\right] = 0
\end{aligned}
\tag{87}
$$

$$
\begin{aligned}
&10\epsilon(t + w) + \epsilon^2\left(w^2 - 1 - \frac{32}{21}\,w' - \frac{12}{7}\,t'\right) + \frac{8\epsilon^3}{105}\left[-\,\frac{4}{3}(2t + 3w)w' - 3tt'\right. \\
&\left. - \frac{422}{21}\,w'' + \frac{151}{7}\,t''\right] + \frac{16\epsilon^4}{525}\left[\frac{15}{4}\,t' - \frac{2}{3}(2tw - 5)w' + \frac{1}{63}(764t + 873w)w''\right. \\
&+ \frac{151}{21}(t + 3w)t'' + \frac{388}{63}(w')^2 + \frac{275}{9}\,w't' + \frac{151}{21}(t')^2 - \frac{6721}{147}\,t''' \\
&\left. - \frac{13100}{(21)^2}\,w'''\right] = 0
\end{aligned}
\tag{88}
$$

Expand w and t in a series in powers of ϵ

$$
\left.
\begin{aligned}
w &= w_0 + \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 + \ldots \\
t &= t_0 + \epsilon t_1 + \epsilon^2 t_2 + \epsilon^3 t_3 + \ldots
\end{aligned}
\right\}
\tag{89}
$$

so that

$$
w^2 = w_0{}^2 + 2\epsilon w_0 w_1 + \epsilon^2\left(w_1{}^2 + 2w_0 w_2\right) + \ldots
$$

$$
tw' = t_0 w_0{}' + \epsilon\left(t_0 w_1{}' + t_1 w_0{}'\right) + \ldots
$$

and so forth. Substituting equations (89) into equations (87) and (88) and arranging terms in increasing powers of ϵ up to ϵ^4 give two very long equations which will not be written out since the next step consists in setting each coefficient of the powers of ϵ equal to zero. The final result is a pair of differential equations for each power of ϵ as follows:

$$t_0 + w_0 = 0 \tag{90}$$

$$t_1 + w_1 + \frac{1}{2}\left(w_0^2 - 1\right) - \frac{8}{21} w_0' = 0 \tag{91}$$

$$5\left(t_1 + w_1\right) + \frac{1}{2}\left(w_0^2 - 1\right) - \frac{16}{21} w_0' - \frac{6}{7} t_0' = 0 \tag{92}$$

$$t_2 + w_2 + w_0 w_1 - \frac{8}{21} w_1' - \frac{16}{315} t_0 w_0' - \frac{8}{105} w_0 w_0' + \frac{32}{(21)^2} w_0'' + \frac{32}{735} t_0'' = 0 \tag{93}$$

$$5\left(t_2 + w_2\right) + w_0 w_1 - \frac{16}{21} w_1' - \frac{6}{7} t_1' - \frac{32}{315} t_0 w_0' - \frac{16}{105} w_0 w_0' - \frac{4}{35} t_0 t_0'$$
$$+ \frac{1688}{2205} w_0'' + \frac{604}{735} t_0'' = 0 \tag{94}$$

$$t_3 + w_3 + \frac{1}{2} w_1^2 + w_0 w_2 - \frac{8}{21} w_2' - \frac{16}{315}\left(t_0 w_1' + t_1 w_0'\right) - \frac{8}{105}\left(w_0 w_1' + w_1 w_0'\right)$$
$$+ \frac{32}{(21)^2} w_1'' + \frac{32}{735} t_1'' + \frac{8}{1575}\left(5 - 2 t_0 w_0\right) w_0' + \frac{32}{2205}\left(3 w_0 + \frac{2}{3} t_0\right) w_0''$$
$$+ \frac{64}{(105)^2}\left(3 w_0 + t_0\right) t_0'' + \frac{32}{945}\left(w_0'\right)^2 + \frac{64}{6615} w_0' t_0' + \frac{64}{(105)^2}\left(t_0'\right)^2$$
$$- \frac{512}{(21)^3} w_0''' - \frac{704}{35(21)^2} t_0''' = 0 \tag{95}$$

$$5\left(t_3 + w_3\right) + \frac{1}{2} w_1^2 + w_0 w_2 - \frac{16}{21} w_2' - \frac{6}{7} t_2' - \frac{32}{315}\left(t_0 w_1' + t_1 w_0'\right)$$
$$- \frac{16}{105}\left(w_0 w_1' + w_1 w_0'\right) - \frac{4}{35}\left(t_0 t_1' + t_1 t_0'\right) + \frac{1688}{5(21)^2} w_1'' + \frac{604}{735} t_1''$$
$$+ \frac{2}{35} t_0' - \frac{32}{1575} t_0 w_0 w_0' + \frac{16}{315} w_0' + \frac{8}{525}\left[\frac{764}{63} t_0 w_0'' + \frac{97}{7} w_0 w_0''\right.$$
$$+ \frac{151}{21} t_0 t_0'' + \frac{151}{7} w_0 t_0'' + \frac{388}{63}\left(w_0'\right)^2 + \frac{275}{9} w_0' t_0' + \frac{151}{21}\left(t_0'\right)^2$$
$$\left. - \frac{6721}{147} t_0''' - \frac{13100}{(21)^2} w_0'''\right] = 0 \tag{96}$$

The solution is found step by step. From equation (90) $t_0 = -w_0$, which when substituted into equation (92) and $t_1 + w_1$ is eliminated between equations (91) and (92) gives for w_0 the differential equation

$$w_0{}' \equiv \frac{dw_0}{d\eta} = w_0{}^2 - 1 \tag{97}$$

The solution is

$$w_0 = -t_0 = -\tanh \eta \tag{98}$$

and equation (91) or (92) becomes

$$t_1 + w_1 = -\frac{5}{42}\left(w_0{}^2 - 1\right) \tag{99}$$

Eliminate $t_2 + w_2$ between equations (93) and (94), obtaining with the aid of equations (97) to (99) a differential equation for w_1

$$\frac{dw_1}{d\eta} - 2w_0w_1 = \frac{29}{245}\, w_0\left(w_0{}^2 - 1\right)$$

whose solution is

$$w_1 = -\frac{29}{490}\left(1 - w_0{}^2\right)\log_e\left(1 - w_0{}^2\right) \tag{100}$$

Then from equation (99)

$$t_1 = \left(1 - w_0{}^2\right)\left[\frac{5}{42} + \frac{29}{490}\,\log_e\left(1 - w_0{}^2\right)\right] \tag{101}$$

So far the results are the same as those the Burnett equations gave for Talbot and Sherman. In fact, continuing in this manner, the form of the solution for w_2 is also the same, being

$$w_2 = \left(1 - w_0{}^2\right)\left\{-A\eta + Bw_0 + Cw_0\,\log_e\left(1 - w_0{}^2\right) + Dw_0\left[\log_e\left(1 - w_0{}^2\right)\right]^2\right\} \tag{102}$$

However, the values of two of the constants A and B are different. The constants are

$$\left.\begin{aligned}
A &= \frac{422}{(245)^2} - \left(\frac{29}{245}\right)^2 = -0.00698042 \\
B &= \frac{576}{49(245)} - A = 0.0549604 \\
C &= -\frac{1}{2}\left(\frac{29}{245}\right)^2 = -0.00700541 \\
D &= -\left(\frac{29}{490}\right)^2 = -0.00350270
\end{aligned}\right\} \tag{103}$$

Substitute equations (100) and (102) into equation (89) and the solution for w becomes

$$w = w_0 - \epsilon\frac{29}{490}\left(1 - w_0^2\right)\log_e\left(1 - w_0^2\right) + \epsilon^2\left(1 - w_0^2\right)\left\{-A\eta + Bw_0 \right.$$
$$\left. + Cw_0 \log_e\left(1 - w_0^2\right) + Dw_0\left[\log_e\left(1 - w_0^2\right)\right]^2\right\} \tag{104}$$

This can now be used to find the thickness of the shock wave, which will be taken to be the maximum slope thickness defined by

$$\delta_m{}^w(\eta) = \frac{w_f - w_i}{\left(\frac{dw}{d\eta}\right)_{max}} \tag{105}$$

Now $w_f - w_i = -2$ while $\frac{dw}{d\eta}$ is a maximum where $\frac{d^2w}{d\eta^2} = 0$. Differentiating equation (104) twice, the expression for $\frac{d^2w}{d\eta^2}$ is found as a series in ϵ to terms in ϵ^2. Since the equation $\frac{d^2w}{d\eta^2} = 0$ need not be solved exactly and probably cannot be solved so, the solution is taken to be

$$\eta = Y = Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \ . \ . \ . \tag{106}$$

Taking $Y_0 = 0$, which simply fixes the origin, and using the infinite series for the hyperbolic tangent

$$w_0 = -\tanh Y = -\epsilon Y_1 - \epsilon^2 Y_2 + \frac{1}{3}\left(\epsilon Y_1 + \epsilon^2 Y_2\right)^3 - \ldots$$

$$= -\epsilon Y_1 - \epsilon^2 Y_2 + O\left(\epsilon^3\right)$$

this value of w_0 is substituted into the expression for $\dfrac{d^2w}{d\eta^2}$. The result is to terms in ϵ^2

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$$\left(\frac{d^2w}{d\eta^2}\right)_{\eta=Y} = 2\epsilon Y_1 + 2\epsilon^2 Y_2 + \epsilon\,\frac{29}{245} + O\left(\epsilon^3\right) = 0$$

with the solution

$$Y_1 = -\frac{29}{490}$$

$$Y_2 = 0$$

Then

$$\left(\frac{dw}{d\eta}\right)_{max} = -1 + \kappa\epsilon^2$$

where

$$\kappa = Y_1{}^2 + \frac{29}{245}\,Y_1 - A - B = -0.051483$$

Using y instead of η as the dimensionless coordinate, the velocity thickness of the shock wave is

$$\delta_m{}^w(y) = \frac{w_f - w_i}{\dfrac{d\eta}{dy}\left(\dfrac{dw}{d\eta}\right)_{max}}$$

Using the values just calculated for $w_f - w_i$, $\left(\dfrac{dw}{d\eta}\right)_{max}$, and the derivative $\dfrac{d\eta}{dy}$ from equation (84)

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$$\delta_m^W(y) = \frac{-2}{-\left(1 - \kappa\epsilon^2\right)\left(1 - \frac{\epsilon^2}{25}\right)} = 2\left[1 - \left(\kappa + \frac{1}{25}\right)\epsilon^2 + \ldots\right]^{-1}$$

$$= 2\left(1 - 0.011483\epsilon^2 + \ldots\right) \tag{107}$$

This agrees best with the result in reference 10 for a Navier-Stokes profile for a Maxwell gas, which in the present notation becomes

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$$\delta_m^W(y) = 2\left(1 - 0.01518\epsilon^2 + \ldots\right)$$

In order to compare the present calculated thickness with the results of reference 9 a transformation of equation (107) must be made. Since

$$\delta_m^W(x) = \delta_m^W(y)\left(\frac{dy}{dx}\right)^{-1}$$

and

$$\frac{dy}{dx} = \frac{4}{35}\frac{\epsilon}{L_*}$$

$$\delta_m^W(x) = \frac{35L_*}{2\epsilon}\left(1 - 0.011483\epsilon^2\right)$$

or

$$\frac{L_*}{\delta_m^W(x)} = \frac{2\epsilon}{35}\frac{1}{1 - 0.011483\epsilon^2} \tag{108}$$

with ϵ given by equation (71). Calculated values of equation (108) are plotted in figure 1 together with two theoretical curves and the experimental data taken from figure 8 of reference 9 and from reference 12. The data found by the optical-reflectivity method as taken from reference 11 and also plotted in figure 1 do not agree with the same data plotted in figure 8 of reference 9. This may be due to a slightly different estimate of L_*. All three of the theoretical curves are based upon the viscosity law for Maxwell molecules ($\mu \propto T$). It is interesting that the present calculations lead to a result not considerably different from the Navier-Stokes curve.

The theoretical velocity profiles through the shock wave are shown in figure 7 of reference 9 at two initial Mach numbers for the Navier-Stokes, the Burnett, and the 13-moment equations. Figure 2 of the present report reproduces two of these profiles, the Navier-Stokes and the Burnett, for $M_1 = 1.576$ together with points calculated from equation (104) of the present paper. The new points agree best with the Navier-Stokes profile as the flow enters the shock wave and with the Burnett profile as it leaves, but on the whole the agreement is with the Navier-Stokes curve.

Finally, a comparison of the shock-wave thickness in terms of upstream mean free path Λ_1 can be made, where for a Maxwell gas

$$\Lambda_1 = \frac{16}{5}\sqrt{\frac{5}{6\pi}}\,\frac{M_1\mu_1}{\rho u}$$

Since

$$\frac{\Lambda_1}{L_*} = \frac{16}{5}\sqrt{\frac{5}{6\pi}}\,\frac{M_1\mu_1}{\mu_*} = \frac{64}{15}\sqrt{\frac{5}{6\pi}}\,\frac{M_1}{1 + \frac{1}{3}M_1^2} \tag{109}$$

$\Lambda_1 \big/ \delta_m{}^w(x)$ is the product of equation (108) with equation (109). Figure 3 is a comparison of the present calculation with the experimental data and two theoretical curves taken from figure 9 of reference 9 and from references 11 and 12. The agreement is again similar to that of figure 1.

The agreement of the present calculation with the Navier-Stokes theory and with the experimental data is probably fortuitous, although the series method of calculation does work better for the present method than it does for the Burnett and 13-moment equations. All of these approaches are essentially based upon the assumption that deviation of the flow from equilibrium is small and that the gradients of velocity, temperature, and pressure are not large. The series expansion breaks down, of course, if ϵ becomes too large and although $\epsilon \to 3$ as $M_1 \to \infty$, one cannot be too sure of the convergence of the series for $\epsilon \geqq 1$, which occurs when M_1 is of the order of 1.3 to 1.4. For the profile shown in figure 2, $\epsilon = 1.44$, which is already large. This difficulty can be overcome by a numerical calculation of the same type as Sherman performed for the Burnett equations. To integrate numerically the equations of Maxwelliam iteration, or even the exact equations of transfer, for this simplest case of one-dimensional flow would not be much more difficult than for the Burnett equations. It also seems that a type of iteration, not in powers of μ/p, but of the method earlier called Truesdell iteration, might be devised for the shock-wave problem as well.

CONCLUDING REMARKS

Most of the present paper has been devoted to an exposition of the Ikenberry-Truesdell theory of the ideal monatomic gas. For continuum and near-continuum flows this theory, following the ideas of Maxwell and Grad, replaces the Boltzmann equation of classical kinetic theory by an infinite number of partial differential equations in the moments of the distribution function. Since attempts to solve the full Boltzmann equation have been hopeless, it would seem that the solution of the problem is thus made more accessible. However, the solution of an infinite system of differential equations may not be any easier, although it presents a different sort of problem which could give useful and rewarding results.

The equations of transfer are independent of the form of molecular distribution function. The latest research on boundary conditions shows how a knowledge of the distribution function is necessary to formulate these conditions. Hence, any properly formulated boundary conditions could be combined with the equations of transfer for use in boundary value problems. However, the success of the Navier-Stokes equations as the basis of the solution of problems in slip flow discourages anyone in the use of the more complicated equations of Ikenberry and Truesdell. More physical insight into such problems seems to be the present need.

In order to obtain equations which do not require any more knowledge of boundary values than those of the known physical variables of state, the method of Maxwellian iteration was presented in some detail. Another purpose for presenting this method of iteration was the correction of a few errors in the original paper. The application of the resulting equations was made to as simple a flow problem as could be formulated and this was the shock-wave problem. The results did not indicate any superiority of the new equations over the Navier-Stokes equations although this is probably because a series type of solution was made. Another form of solution of this problem or of some other problem should be found before the theory is abandoned and might offer a fertile field for theoretical research.

University of Washington,
Seattle, Wash., July 30, 1958.

APPENDIX A

RELATIONS BETWEEN MOMENTS AND SPHERICAL MOMENTS

Equation (5) of the text defines the moments

$$p_{i_1 i_2 \ldots i_s} = \rho\overline{c_{i_1} c_{i_2} \ . \ . \ . \ c_{i_s}} = \rho\overline{Q}_s = m \int_\infty Q_s F \, d\vec{\xi} \tag{A1}$$

while the spherical moments are given by equation (22)

$$P_{2r|s} = \rho\overline{Y}_{2r|s} = m \int_\infty c^{2r} Y_s(\vec{c}) F \, d\vec{c} \tag{A2}$$

Taking $Q = 1, c_i, \ . \ . \ . \ ,$ equation (A1) gives, using the fact that $d\vec{\xi} = d\vec{c}$ in velocity space:

$Q = 1$:

$$\rho = m \int_\infty F \, d\vec{\xi} = P_{0|0} = P \tag{A3}$$

since the subscript r or s will not be written down when $r = 0$ or $s = 0$.

$Q = c_i$:

$$\rho\bar{c}_i = m \int_\infty c_i F \, d\vec{\xi} = P_i = 0 \tag{A4}$$

$Q = c_i c_j$:

$$\rho\overline{c_i c_j} = p_{ij} = m \int_\infty c_i c_j F \, d\vec{\xi} = m \int_\infty Y_{ij} F \, d\vec{c} + \frac{1}{3} \delta_{ij} m \int_\infty c^2 F \, d\vec{c}$$

from the definition of Y_{ij} in equation (20). Then equations (A2) and (8) give

$$p_{ij} = P_{ij} + p\delta_{ij} \tag{A5}$$

which is the same as equation (10). The contracted form is

$$p_{ii} = 3p = P_{2|} \tag{A6}$$

since $Y_{ii} = 0$.

$Q = c_i c_j c_k$:

$$p_{ijk} = m\int_\infty c_i c_j c_k F \, d\vec{\xi} = m\int_\infty Y_{ijk}F \, d\vec{c} + \frac{3}{5}\, m\delta_{(ij}\int_\infty c^2 c_{k)} F \, d\vec{c}$$

$$= P_{ijk} + \frac{3}{5} P_{2|(i}\delta_{jk)} \tag{A7}$$

with the contraction

$$p_{ijj} = P_{2|i} = 2q_i \tag{A8}$$

since $P_{ijj} = 0$ and

$$3P_{2|(i}\delta_{jj)} = P_{2|i}\delta_{jj} + P_{2|j}\delta_{ij} + P_{2|j}\delta_{ij} = 5P_{2|i}$$

while $P_{2|i} = m\int_\infty c^2 c_i F \, d\vec{c} = 2q_i$ by equation (7).

$Q = c_i c_j c_k c_l$:

$$p_{ijkl} = m\int_\infty c_i c_j c_k c_l F \, d\vec{\xi}$$

$$= m\int_\infty Y_{ijkl}F \, d\vec{c} + \frac{6}{7}\, m\int_\infty c^2 Y_{(ij}\delta_{kl)} F \, d\vec{c} + \frac{1}{5}\, m\int_\infty c^4 \delta_{(ij}\delta_{kl)} F \, d\vec{c}$$

$$= P_{ijkl} + \frac{6}{7} P_{2|(ij}\delta_{kl)} + \frac{1}{5} P_{4|}\delta_{(ij}\delta_{kl)} \tag{A9}$$

with the contractions

$$p_{ij\ell\ell} = m\int_{\infty} c^2 c_i c_j F \, d\vec{\xi} = m\int_{\infty} c^2 Y_{ij} F \, d\vec{c} + \frac{1}{3}\,\delta_{ij} m\int_{\infty} c^4 F \, d\vec{c}$$

$$= P_{2|ij} + \frac{1}{3}\,P_{4|}\,\delta_{ij} \tag{A10}$$

$$p_{mm\ell\ell} = m\int_{\infty} c^4 F \, d\vec{\xi} = P_{4|} \tag{A11}$$

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$Q = c_i c_j c_k c_\ell c_m$:

$$p_{ijk\ell m} = m\int_{\infty} c_i c_j c_k c_\ell c_m F \, d\vec{\xi}$$

$$= m\int_{\infty} Y_{ijk\ell m} F \, d\vec{c} + \frac{10}{9}\, m\int_{\infty} c^2 Y_{(ijk}\delta_{\ell m)} F \, d\vec{c} + \frac{3}{7}\,\delta_{(ik}\delta_{\ell m} m\int_{\infty} c^4 Y_{i)} F \, d\vec{c}$$

$$= P_{ijk\ell m} + \frac{10}{9}\, P_{2|(ijk}\delta_{\ell m)} + \frac{3}{7}\, P_{4|(i}\delta_{jk}\delta_{\ell m)} \tag{A12}$$

with the contractions

$$p_{ijk\ell\ell} = m\int_{\infty} c^2 c_i c_j c_k F \, d\vec{\xi} = m\int_{\infty} c^2 Y_{ijk} F \, d\vec{c} + \frac{3}{5}\, m\delta_{(jk}\int_{\infty} c^4 c_{i)} F \, d\vec{c}$$

$$= P_{2|ijk} + \frac{3}{5}\, P_{4|(i}\delta_{jk)} \tag{A13}$$

$$p_{ikk\ell\ell} = m\int_{\infty} c^4 c_i F \, d\vec{\xi} = P_{4|i} \tag{A14}$$

Then

$$P_{ijk\ell m} = p_{ijk\ell m} - \frac{10}{9}\, p_{rr(ijk}\delta_{\ell m)} + \frac{5}{21}\, p_{rrss(i}\delta_{jk}\delta_{\ell m)} \tag{A15}$$

$Q = c^4 c_i c_j$:

$$p_{ijllmm} = m \int_\infty c^4 c_i c_j F \, d\vec{\xi} = m \int_\infty c^4 Y_{ij} F \, d\vec{c} + \frac{1}{3} \delta_{ij} m \int_\infty c^6 F \, d\vec{c}$$

$$= P_{4|ij} + \frac{1}{3} P_{6|} \delta_{ij} \tag{A16}$$

with the contraction

$$p_{kkllmm} = P_{6|} \tag{A17}$$

These are all of the relations used in this paper.

APPENDIX B

SAMPLE CALCULATIONS OF $L\left(Y_{2r|s}\right)$ AND $C\left(Y_{2r|s}\right)$

To illustrate the technique of evaluating $L\left(Y_{2r|s}\right)$ and $C\left(Y_{2r|s}\right)$ the calculations are carried out in detail for $Y_{2r|s} = c^4 Y_i$.

Expression (12) for $L(Q)$ with $Q = c^4 Y_i = c^4 c_i$ becomes in this case

$$L\left(c^4 Y_i\right) = \left(\rho\overline{c^4 Y_i}\right)^{\cdot} + \rho\overline{c^4 Y_i}E + \overline{\rho c_k \frac{\partial}{\partial c_j}\left(c^4 c_i\right)}u_{j,k} - \frac{1}{\rho}\, p_{jk,k}\rho\, \overline{\frac{\partial}{\partial c_j}\left(c^4 c_i\right)}$$

$$+ \frac{\partial}{\partial x_j}\left(\rho\overline{c_j c_i c^4}\right) \tag{B1}$$

The averages are given by equations (5) or (22) so

$$\rho\overline{c^4 Y_i} = m\int_\infty c^4 Y_i F \, d\vec{c} = P_{4|i}$$

and

$$u_{j,k}\overline{\rho c_k \frac{\partial}{\partial c_j}\left(c^4 c_i\right)} = u_{i,k}m\int_\infty c^4 Y_k F \, d\vec{c} + 4u_{j,k}m\int_\infty c^2 c_i c_j c_k F \, d\vec{c}$$

$$= u_{i,k}P_{4|k} + 4u_{j,k}\left(P_{2|ijk} + \frac{3}{5}\, P_{4|(i}\delta_{jk)}\right)$$

$$= u_{i,k}P_{4|k} + 2P_{2|ijk}\left(u_{j,k} + u_{k,j}\right) + \frac{12}{5}\, P_{4|(i}\delta_{jk)}u_{j,k}$$

using equation (A13). Now

$$u_{j,k} + u_{k,j} = 2E_{jk} + \frac{2}{3}\, E\delta_{jk}$$

so

$$2P_{2|ijk}\left(u_{j,k} + u_{k,j}\right) = 4P_{2|ijk}E_{jk}$$

and

$$3P_{4|(i}\delta_{jk)}u_{j,k} = EP_{4|i} + P_{4|j}\left(u_{i,j} + u_{j,i}\right) = \frac{5}{3}\,EP_{4|i} + 2P_{4|j}E_{ij}$$

so

$$u_{j,k}\,\overline{\rho c_k \frac{\partial}{\partial c_j}\left(c^4 c_i\right)} = P_{4|j}\left(u_{i,j}\right) + \frac{8}{5}\,E_{ij} + \frac{4}{3}\,EP_{4|i} + 4P_{2|ijk}E_{jk}$$

Then

$$-\frac{1}{\rho}\,p_{jk,k}\rho\,\overline{\frac{\partial}{\partial c_j}\left(c^4 c_i\right)} = -\frac{1}{\rho}\,p_{ik,k}m\int_\infty c^4 F\,d\vec{c} - \frac{4}{\rho}\,p_{jk,k}m\int_\infty c^2 c_i c_j F\,d\vec{c}$$

$$= -\frac{1}{\rho}\,p_{ik,k}P_{4|} - \frac{4}{\rho}\,p_{jk,k}m\int_\infty\left(c^2 Y_{ij} + \frac{1}{3}\,c^4\delta_{ij}\right)F\,d\vec{c}$$

$$= -\frac{7}{3\rho}\,p_{ik,k}P_{4|} - \frac{4}{\rho}\,p_{jk,k}P_{2|ij}$$

and

$$\overline{\rho c_j c_i c^4} = m\int_\infty c^4 c_i c_j F\,d\vec{c} = m\int_\infty\left(c^4 Y_{ij} + \frac{1}{3}\,c^6\delta_{ij}\right)F\,d\vec{c}$$

$$= P_{4|ij} + \frac{1}{3}\,P_{6|}\delta_{ij}$$

Hence

$$L\left(c^4 Y_i\right) = \dot{P}_{4|i} + EP_{4|i} + u_{i,k}P_{4|k} + \frac{8}{5}\,E_{ij}P_{4|j} + \frac{4}{3}\,EP_{4|i} + 4P_{2|ijk}E_{jk}$$

$$-\frac{7}{3\rho}\,p_{ij,j}P_{4|} - \frac{4}{\rho}\,p_{jk,k}P_{2|ij} + P_{4|ij,j} + \frac{1}{3}\,P_{6|,i} \qquad \text{(B2)}$$

which is the last one of equations (25) in the text.

The calculation of $C\left(Y_{2r|s}\right)$ is not so simple since it involves the evaluation of the four integrals in equation (13). In a long section toward the end of chapter 1 of reference 6, Ikenberry and Truesdell show that this collision integral can be expressed exactly as a polynomial in the moments, provided the molecular model is that of Maxwell molecules. In the process they obtain certain simple expressions for these integrals which will now be used to evaluate $C\left(c^4 Y_i\right)$.

Since $Y_i = c_i$ and $Q = c^4 Y_i = c^4 c_i$, in this case $\delta Q = (c^{*\prime})^4 c_i^{*\prime} + (c')^4 c_i' - (c^*)^4 c_i^* - c^4 c_i$ according to equation (17). Equations (18) and (19) give

$$\vec{c} = \vec{w} - \vec{v}$$

$$\vec{c}^{\,*} = \vec{w} + \vec{v}$$

$$\vec{c}\,' = \vec{w} - \vec{v}\,'$$

$$\vec{c}^{\,*\prime} = \vec{w} + \vec{v}\,'$$

so

$$(c^*)^4 + c^4 = 2\left(v^2 + w^2\right)^2 + 8(\vec{v} \cdot \vec{w})^2$$

$$(c^*)^4 - c^4 = 8\left(v^2 + w^2\right)(\vec{v} \cdot \vec{w})$$

and

$$\delta Q = \left[(c^{*\prime})^4 - (c')^4\right]v_i' - \left[(c^*)^4 - c^4\right]v_i + \left[(c^{*\prime})^4 + (c')^4 - (c^*)^4 - c^4\right]w_i$$

$$= 8\left(v^2 + w^2\right)\left[(\vec{v}\,' \cdot \vec{w})v_i' - (\vec{v} \cdot \vec{w})v_i\right] + 8\left[(\vec{v}\,' \cdot \vec{w})^2 - (\vec{v} \cdot \vec{w})^2\right]w_i \tag{B3}$$

using the fact that $(v')^2 = v^2$.

Introducing the spherical harmonic

$$Y_{ij}(\vec{v}) = v_i v_j - \frac{1}{3} v^2 \delta_{ij}$$

the mean rate of change in Q becomes with $\vec{v} \cdot \vec{w} = v_j w_j$

$$\delta Q = 8\left(v^2 + w^2\right)w_j\left(v_i' v_j' - v_i v_j\right) + 8 w_i w_j w_k\left(v_k' v_j' - v_k v_j\right)$$

$$= 8\left(v^2 + w^2\right)w_j\left[Y_{ij}(\vec{v}\,') - Y_{ij}(\vec{v})\right] + 8 w_i w_j w_k\left[Y_{jk}(\vec{v}\,') - Y_{jk}(\vec{v})\right] \tag{B4}$$

The integrals (13) for $C(Q)$ can be split into the four integrals

$$C(Q) = \int_{\infty} F \, d\vec{\xi} \int_{\infty} F^* d\vec{\xi}^* \int_0^{\infty} vb \; db \int_0^{2\pi} \delta Q \; d\epsilon$$

Equation (7.1) of reference 6 gives a formula for the integral over ϵ of the spherical harmonic $Y_s(\vec{v}\,')$ which is

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$$\int_0^{2\pi} Y_s(\vec{v}\,')d\epsilon = 2\pi Y_s(\vec{v})P_s(\cos \phi)$$

with $P_s(\cos \phi)$ being the Legendre polynomial and $\phi = \pi - 2\theta$. Hence

$$\int_0^{2\pi} \delta Q \; d\epsilon = 16\pi\left(v^2 + w^2\right)w_j\left[P_2(\cos \phi) - 1\right]Y_{ij}(\vec{v}) + 16\pi w_i w_j w_k\left[P_2(\cos \phi) - 1\right]Y_{jk}(\vec{v}) \tag{B5}$$

For the next integral the formula is a special case of equation (7.6) of reference 6 or

$$B_2 = 2\pi v \int_0^{\pi} \left[1 - P_2(\cos \phi)\right] b \, \frac{db}{d\phi} \, d\phi$$

so

$$\begin{aligned}\int_0^{\infty} vb \; db \int_0^{2\pi} \delta Q \; d\epsilon &= -8B_2\left[\left(v^2 + w^2\right)w_j Y_{ij}(\vec{v}) + w_i w_j w_k Y_{jk}(\vec{v})\right] \\ &= -8B_2\left[\left(v^2 + w^2\right)(\vec{v} \cdot \vec{w})v_i - \frac{1}{3} v^2\left(v^2 + w^2\right)w_i \right. \\ &\qquad \left. + (\vec{v} \cdot \vec{w})^2 w_i - \frac{1}{3} v^2 w^2 w_i\right]\end{aligned} \tag{B6}$$

The last two integrations over $d\vec{\xi} = d\vec{c}$ and $d\vec{\xi}^* = d\vec{c}^*$ require a return to the original variables c and c^*. Using the identities

$$(c^*)^4 - c^4 = 8\left(v^2 + w^2\right)(\vec{v} \cdot \vec{w})$$

$$(c^*)^2 + c^2 = 2\left(v^2 + w^2\right)$$

$$4v^2 = (c^*)^2 - 2(\vec{c}^* \cdot \vec{c}) + c^2$$

$$2v_i = c_i{}^* - c_i$$

$$2w_i = c_i{}^* + c_i$$

$$16(\vec{v} \cdot \vec{w})^2 = (c^*)^4 - 2(c^*)^2c^2 + c^4$$

$$4w^2 = (c^*)^2 + 2(\vec{c}^* \cdot \vec{c}) + c^2$$

$$16v^2w^2 = (c^*)^4 - 4(\vec{c}^* \cdot \vec{c})^2 + 2c^2(c^*)^2 + c^4$$

after simplifying and collecting terms integral (B6) becomes

$$\begin{aligned}\int_0^\infty vb\ db \int_0^{2\pi} \delta Q\ d\epsilon = -B_2\Big[\tfrac{1}{2}(c^*)^4\left(c_i{}^* - c_i\right) &+ \tfrac{1}{2}\, c^4\left(c_i - c_i{}^*\right) \\ - c^2(c^*)^2\left(c_i{}^* + c_i\right) &+ \tfrac{1}{3}\, c^2 c_j c_j{}^*\left(c_i{}^* + c_i\right) \\ + \tfrac{1}{3}(c^*)^2 c_j{}^* c_j\left(c_i{}^* + c_i\right) &+ \tfrac{1}{3}\, c_j{}^* c_k{}^* c_j c_k\left(c_i{}^* + c_i\right)\Big]\end{aligned} \tag{B7}$$

Multiplying this result by $m^2 FF^*$ and integrating over $\vec{c}$ and $\vec{c}^*$ gives, since

$$m\int_\infty (c^*)^4 c_i{}^* F^* d\vec{c}^* = m\int_\infty c^4 c_i F\ d\vec{c} = P_{4|i}$$

$$m\int_\infty c_i{}^* F^* d\vec{c}^* = m\int_\infty c_i F\ d\vec{c} = 0$$

$$m\int_\infty F^* d\vec{c}^* = m\int_\infty F\ d\vec{c} = \rho$$

$$m\int_\infty (c^*)^2 c_i{}^* F^* d\vec{c}^* = m\int_\infty c^2 c_i F\ d\vec{c} = P_{2|i} = 2q_i$$

$$m\int_\infty (c^*)^2F^*d\vec{c}^* = m\int_\infty c^2F\,d\vec{c} = P_{2|} = 3p$$

$$m\int_\infty c_i^*c_j^*F^*d\vec{c}^* = m\int_\infty c_ic_jF\,d\vec{c} = p_{ij} = P_{ij} + p\delta_{ij}$$

$$m\int_\infty c_i^*c_j^*c_k^*F^*d\vec{c}^* = m\int_\infty c_ic_jc_kF\,d\vec{c} = p_{ijk} = P_{ijk} + \frac{6}{5}\,q_{(i}\delta_{jk)}$$

the collision integral

$$mC(Q) = -\frac{B_2}{m}\left(\rho P_{4|i} - 12pq_i + \frac{4}{3}\,p_{ij}q_j + \frac{2}{3}\,p_{ijk}p_{jk}\right)$$

Because $P_{ijj} = P_{jj} = 0$, $\delta_{jk}\delta_{jk} = 3$, and
$q_{(i}\delta_{jk)} = \frac{1}{3}\left(q_i\delta_{jk} + q_j\delta_{ki} + q_k\delta_{ij}\right)$ this simplifies to

$$mC\left(c^4Y_i\right) = -\frac{nB_2}{\rho}\left(\rho P_{4|i} - \frac{28}{3}\,pq_i + \frac{2}{3}\,P_{ijk}P_{jk} + \frac{28}{15}\,P_{ij}q_j\right)$$

which is, with $nB_2 = \frac{p}{\mu}$, the next to the last of equations (36). It is the corrected form of equation (8.10) of reference 6.

APPENDIX C

USEFUL DERIVATIVES

From $p = R\rho T$ and $\rho u = m$, if the prime indicates the derivative with respect to x,

$$\frac{p'}{p} = \frac{\rho'}{\rho} + \frac{T'}{T}$$

$$\frac{\rho'}{\rho} + \frac{u'}{u} = 0$$

so

$$\frac{p'}{p} = \frac{T'}{T} - \frac{u'}{u}$$

$$\left(\frac{1}{\rho}\right)' = \frac{u'}{\rho u}$$

Also,

$$\left(\frac{1}{p}\right)' = - \frac{T'}{pT} + \frac{u'}{pu}$$

$$\left(\frac{p^2}{\rho}\right)' = 2\,\frac{p^2T'}{\rho T} - \frac{p^2u'}{\rho u}$$

$$\frac{\rho''}{\rho} = 2\,\frac{(u')^2}{u^2} - \frac{u''}{u}$$

$$\frac{p''}{p} = 2\,\frac{(u')^2}{u^2} - \frac{u''}{u} - 2\,\frac{u'T'}{uT} + \frac{T''}{T}$$

The entropy

$$\beta = \frac{3}{2} \log_e \frac{p}{\rho^{5/3}}$$

so

$$\beta' = \frac{3}{2}\frac{T'}{T} + \frac{u'}{u}$$

and

$$\beta'' = \frac{3}{2}\,\frac{T''}{T} - \frac{3}{2}\,\frac{(T')^2}{T^2} + \frac{u''}{u} - \frac{(u')^2}{u^2}$$

The coefficient of viscosity

$$\mu = KT$$

so

W 1 3 4

$$\mu' = \frac{\mu}{T}\,T'$$

Then

$$(\mu u')' = \frac{\mu}{T}\,u'T' + \mu u''$$

$$(\mu T')' = \frac{\mu}{T}(T')^2 + \mu T''$$

$$\left(\frac{\mu}{\rho}\,u'\right)' = \frac{\mu}{\rho T}\,u'T' + \frac{\mu}{\rho u}(u')^2 + \frac{\mu}{\rho}\,u''$$

$$\left(\frac{\mu p}{\rho}\,u'\right)' = 2\,\frac{\mu p}{\rho T}\,u'T' + \frac{\mu p}{\rho}\,u''$$

$$\left(\frac{\mu p}{\rho T}\,uT'\right)' = \frac{\mu p}{\rho T^2}\,u(T')^2 + \frac{\mu p}{\rho T}\,u'T' + \frac{\mu p}{\rho T}\,uT''$$

$$\left(\frac{\mu^2}{\rho}\,u''\right)' = 2\,\frac{\mu^2}{\rho T}\,u''T' + \frac{\mu^2}{\rho u}\,u'u'' + \frac{\mu^2}{\rho}\,u'''$$

$$\left[\frac{\mu^2}{\rho}(u')^2\right]' = 2\,\frac{\mu^2}{\rho T}(u')^2T' + \frac{\mu^2}{\rho u}(u')^3 + 2\,\frac{\mu^2}{\rho}\,u'u''$$

$$\left[\frac{\mu^2}{\rho u}(u')^2\right]' = 2\,\frac{\mu^2}{\rho u T}(u')^2T' + 2\,\frac{\mu^2}{\rho u}\,u'u''$$

$$\left(\frac{\mu^2}{\rho T}\,T''\right)' = \frac{\mu^2}{\rho T^2}\,T'T'' + \frac{\mu^2}{\rho u T}\,u'T'' + \frac{\mu^2}{\rho T}\,T'''$$

$$\left(\frac{\mu^2}{\rho T}\,u'T'\right)' = \frac{\mu^2}{\rho T^2}\,u'(T')^2 + \frac{\mu^2}{\rho u T}(u')^2T' + \frac{\mu^2}{\rho T}\,u''T' + \frac{\mu^2}{\rho T}\,u'T''$$

$$\left(\frac{\mu^2}{\rho T} uT''\right)' = 2 \frac{\mu^2}{\rho T} u'T'' + \frac{\mu^2}{\rho T^2} uT'T'' + \frac{\mu^2}{\rho T} uT'''$$

$$\left(\frac{\mu^2}{\rho} uu''\right)' = 2 \frac{\mu^2}{\rho T} uu''T' + 2 \frac{\mu^2}{\rho} u'u'' + \frac{\mu^2}{\rho} uu'''$$

$$\left[\frac{\mu^2}{\rho T^2}(T')^2\right]' = \frac{\mu^2}{\rho u T^2} u'(T')^2 + 2 \frac{\mu^2}{\rho T^2} T'T''$$

$$\left[\frac{\mu^2}{\rho T^2} u(T')^2\right]' = 2 \frac{\mu^2}{\rho T^2} u'(T')^2 + 2 \frac{\mu^2}{\rho T^2} uT'T''$$

$$\left(\frac{\mu^2}{\rho T} uu'T'\right)' = \frac{\mu^2}{\rho T^2} uu'(T')^2 + 2 \frac{\mu^2}{\rho T}(u')^2 T' + \frac{\mu^2}{\rho T} uu''T' + \frac{\mu^2}{\rho T} uu'T''$$

$$\left(\frac{\mu^2}{\rho T} u^2T''\right)' = \frac{\mu^2}{\rho T^2} u^2T'T'' + 3 \frac{\mu^2}{\rho T} uu'T'' + \frac{\mu^2}{\rho T} u^2T'''$$

$$\left[\frac{\mu^2}{\rho T^2} u^2(T')^2\right]' = \frac{\mu^2}{\rho T^2} uu'(T')^2 + 2 \frac{\mu^2}{\rho T^2} uu'(T')^2 + 2 \frac{\mu^2}{\rho T^2} u^2T'T''$$

$$\left[\frac{\mu^2}{p}(u')^2\right]' = \frac{\mu^2}{pT}(u')^2T' + \frac{\mu^2}{pu}(u')^3 + 2 \frac{\mu^2}{p} u'u''$$

$$\left(\frac{\mu^2}{p} uu''\right)' = \frac{\mu^2}{pT} uu''T' + 2 \frac{\mu^2}{p} u'u'' + \frac{\mu^2}{p} uu'''$$

$$\left(\frac{\mu^2}{pT} uu'T'\right)' = 2 \frac{\mu^2}{pT}(u')^2T' + \frac{\mu^2}{pT} uu''T' + \frac{\mu^2}{pT} uu'T''$$

$$\left(\frac{\mu^2 p}{\rho^2 T} T''\right)' = 2 \frac{\mu^2 p}{\rho^2 T^2} T'T'' + \frac{\mu^2 p}{\rho^2 uT} u'T'' + \frac{\mu^2 p}{\rho^2 T} T'''$$

$$\left[\frac{\mu^2 p}{\rho^2 T^2}(T')^2\right]' = \frac{\mu^2 p}{\rho^2 T^3}(T')^3 + \frac{\mu^2 p}{\rho^2 uT^2} u'(T')^2 + 2 \frac{\mu^2 p}{\rho^2 T^2} T'T''$$

$$\left(\frac{\mu^2 p}{\rho^2 uT} u'T'\right)' = 2 \frac{\mu^2 p}{\rho^2 uT^2} u'(T')^2 + \frac{\mu^2 p}{\rho^2 uT} u''T' + \frac{\mu^2 p}{\rho^2 uT} u'T''$$

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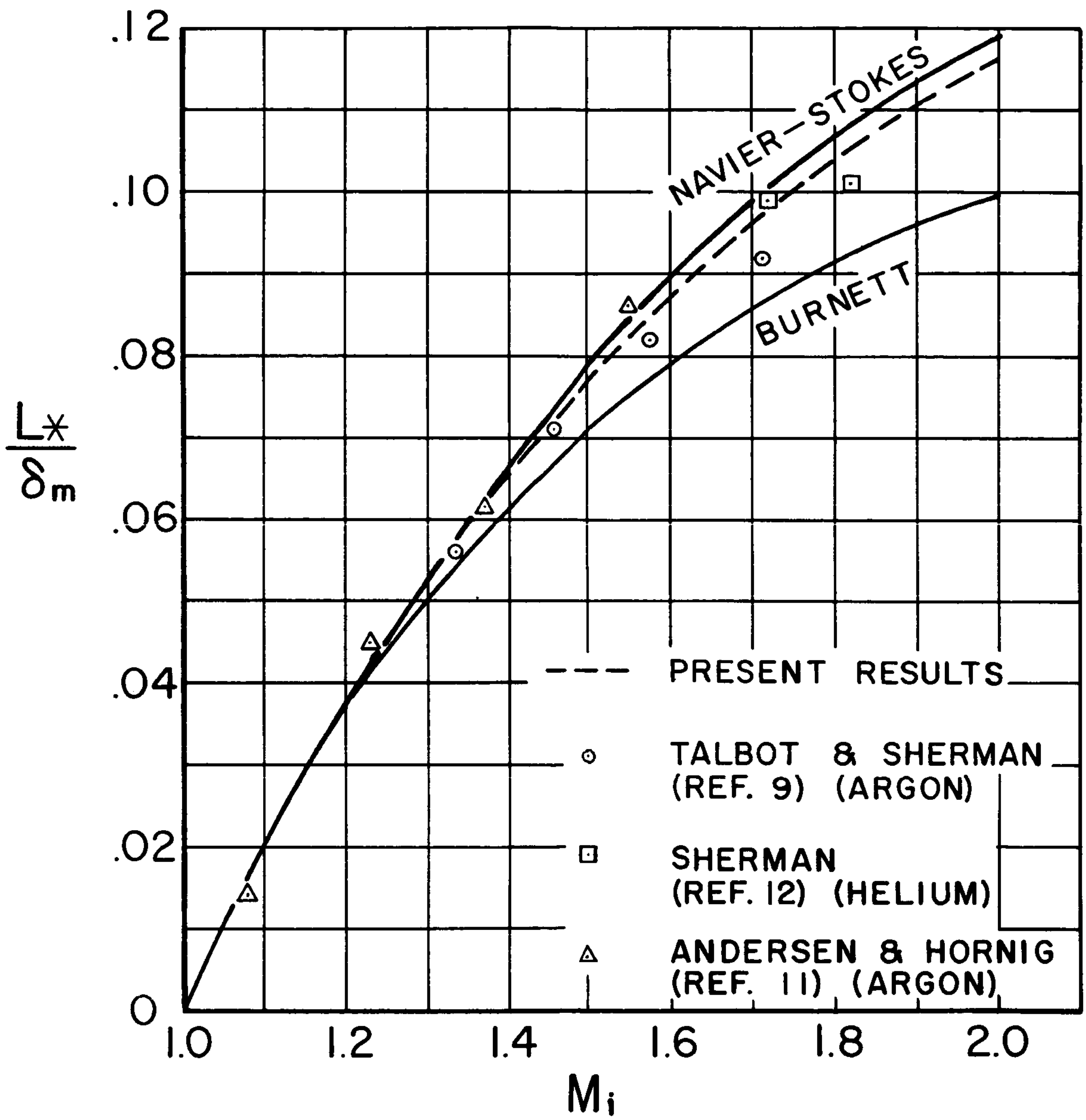

Figure 1.- Shock-wave thickness in terms of reference length L_*.

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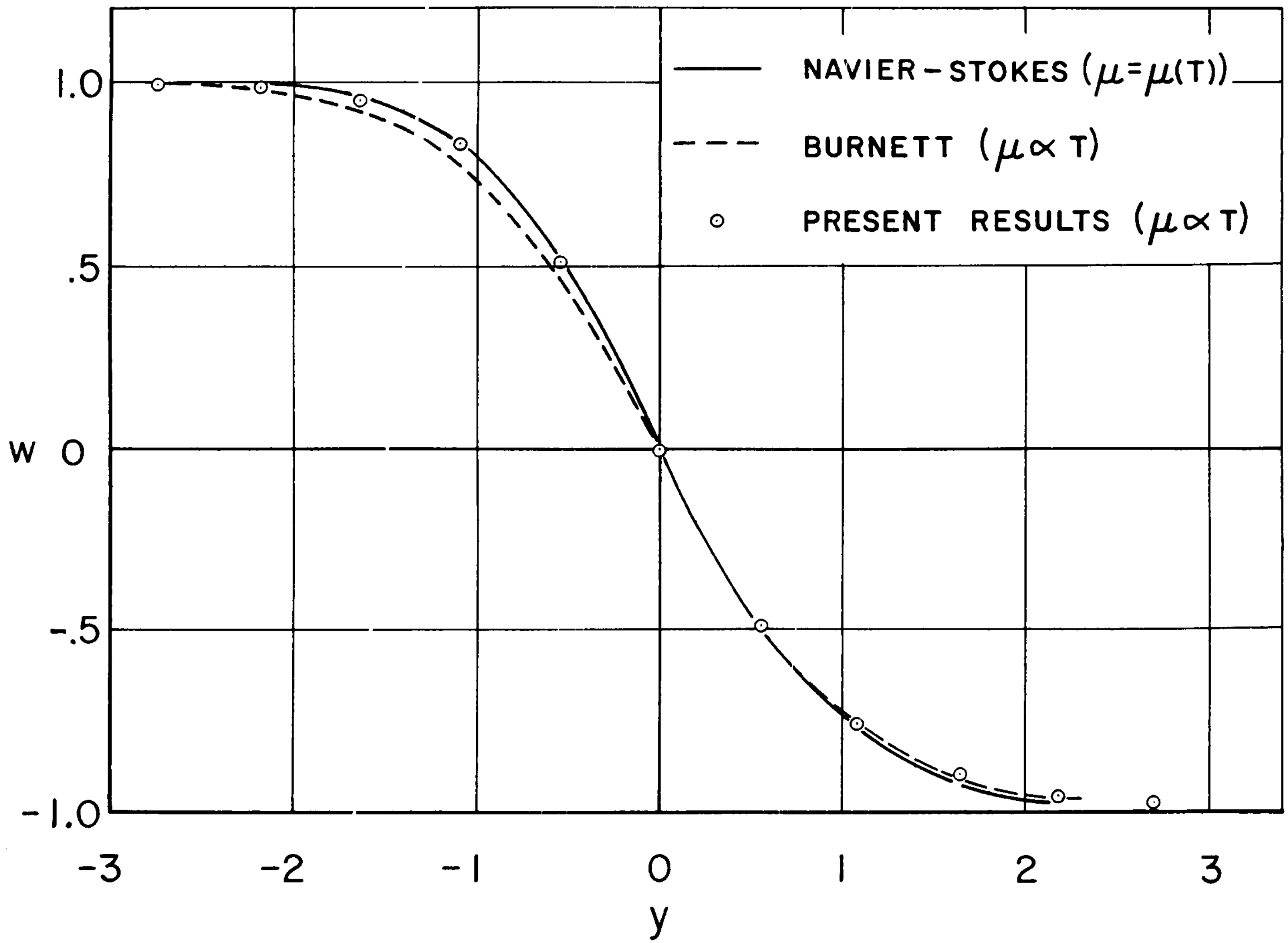

Figure 2.- Velocity profiles. $M_1 = 1.576$.

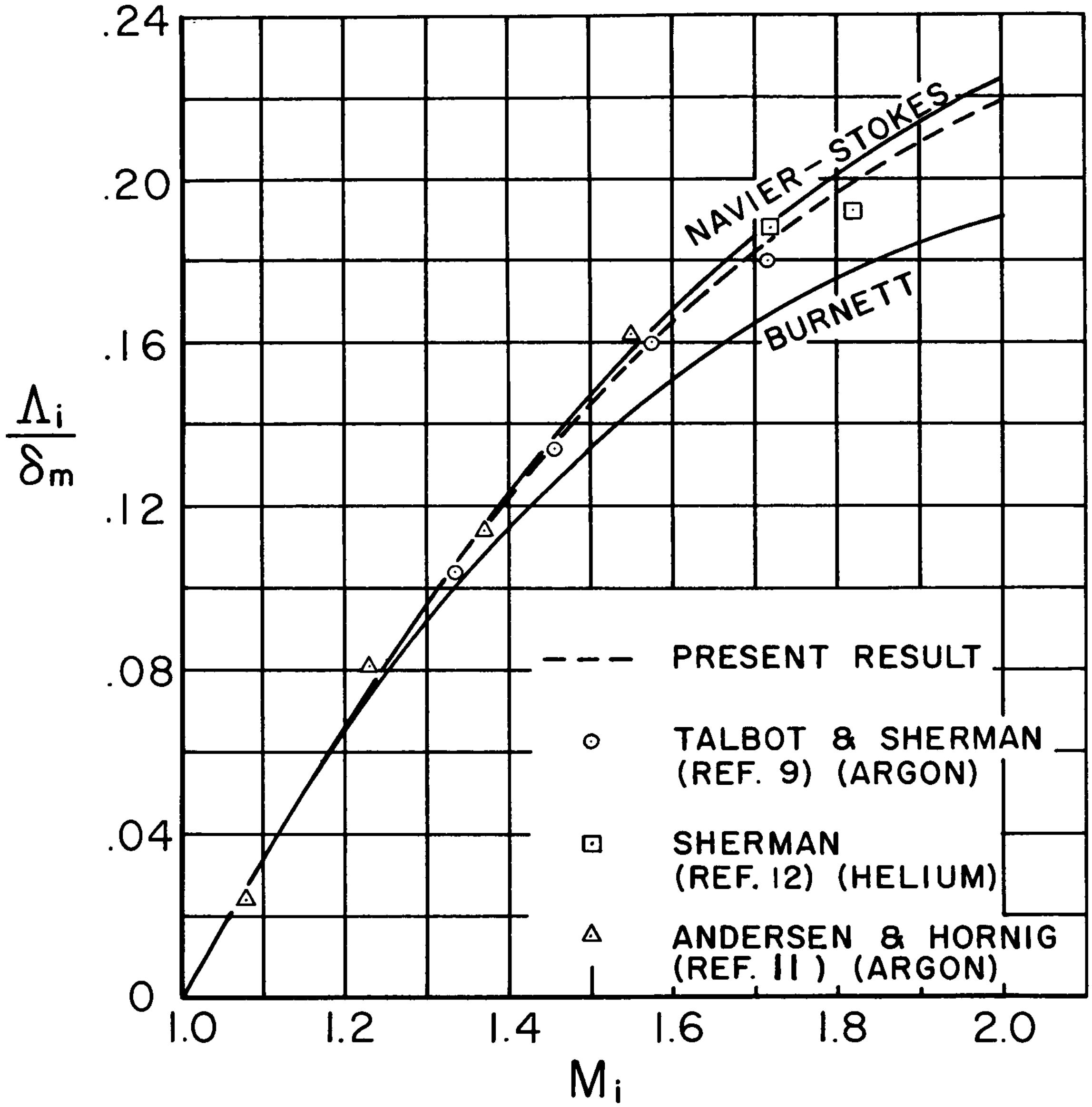

Figure 3.- Shock-wave thickness in terms of upstream mean free path.